1965

A parametric traveling-wave amplifier using oriented thin magnetic film

Juang Chi Chang
Iowa State University

CHANG, Juang Chi, 1936-
A PARAMETRIC TRAVELING-WAVE AMPLIFIER
USING ORIENTED THIN MAGNETIC FILM.

Iowa State University of Science and Technology,
Ph.D., 1965
Engineering electrical

University Microfilms, Inc., Ann Arbor, Michigan

A PARAMETRIC TRAVELING-WAVE AMPLIFIER USING

ORIENTED THIN MAGNETIC FILM

by

Juang Chi Chang

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of

The Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Electrical Engineering

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1965

TABLE OF CONTENTS

I. INTRODUCTION

Parametric amplification is associated with frequency-mixing devices which utilize the properties of nonlinear or time-varying reactances. The function of this reactance is to channel energy from an a-c source to a useful load. In this sense, the parametric amplification is similar to the vacuum tube amplification. The vacuum tube is essentially a variable resistance which converts d-c energy to useful a-c energy. Any resistor at a non-zero temperature exhibits noise properties which places a limit on useful amplification in a conventional amplifier. An essential and very desirable feature of parametric amplification is due to its nonlinear reactance. Since the reactance does not contribute thermal noise to a circuit, the parametric amplifier can have excellent noise performance.

The discovery of the parametric amplifier came much later than the discovery of the basic principles of operation. R. V. L. Hartley[1] in 1936 discussed in great detail an electro-mechanical nonlinear capacitance device very similar to today's negative-resistance parametric amplifier. With the passing of time, various experimental and theoretical papers were published describing circuits with nonlinear reactive elements. An important one among them was the paper by van der Ziel[2] in 1948. In this paper he analyzed mixer circuits containing nonlinear capacitance. In addition, the bandwidth and the noise factor of mixer circuits were discussed. It was shown on theoretical grounds that the circuit might have a very low noise factor.

In 1956, Manley and Rowe[3] published their famous paper concerning the general energy relations of some general properties of nonlinear elements.

Next year, Suhl[4] proposed a microwave solid-state amplifier using ferrite. Shortly later Weiss[5] announced the experimental verification of Suhl's proposal. It was thus almost ten years after the publication of van der Ziel's paper that a parametric amplifier was built. Since then, parametric amplification has gained special interest theoretically and experimentally among many people[6,7]. Most of them used back-biased semiconductor diodes as a nonlinear capacitive reactance for the parametric amplifiers. In 1959, Pohm and Read[8] successfully used a magnetic thin film as a time-varying inductor in a parametric amplifier. Later on, the thin film balanced modulator[9] and the operation of magnetic thin film parametron[10] were also realized.

One limitation of the single tuned parametric amplifier is its restricted bandwidth. Typical bandwidths of experimental single tuned parametric amplifiers at useful values of gain have been on the order of 1%. That is the ratio of the range of frequencies for which the gain of the amplifier falls within one half of its maximum value to the operating maximum gain frequency is on the order of 1%. A second limitation of the amplifier is its potentially unstable and bilateral gain characteristic. Multiple-tuned parametric amplifiers have been designed which have improved bandwidth up to 5%[11]. Nevertheless, a multiple-tuned parametric amplifier is still a potentially unstable device with bilateral gain characteristic.

If the resonant structure of either a single or multiple tuned parametric amplifier is replaced by a suitable traveling-wave propagating structure, it is possible to achieve a measure of unilateral gain with improved stability and bandwidth. Theoretical analyses and experimental investigations have been reported by many people concerning this kind of traveling-wave parametric amplification[12-18]. Typical bandwidths of around 15% with good stability and unilateral characteristic have been obtained. All the experimental traveling-wave parametric amplifiers have used semiconductor diodes as time-varying capacitors. No successful operation of a traveling-wave parametric amplifier using time-varying inductors has been reported yet.

The purpose of this paper is to study theoretically and experimentally the wave propagation in a periodic traveling-wave structure using thin magnetic film as time-varying inductor. The magnetic thin film is electrodeposited along a pump line which is essentially a transmission line with series inductance L_p and shunt capacitance C_p (see Figure 7). Signal and idler waves are traveling along another transmission line structure with series inductance L_s and parallel capacitance C_s. A large part of L_s is contributed by the thin film. The analysis will show that L_s along the signal line is a time-varying function of pump frequency ω_p and it is due to this time-varying inductance that the parametric interaction takes place. The Manley and Rowe relation[3] describes how this parametric interaction will permit a net transfer of a portion of the pump energy into signal energy. Thus it is possible to obtain power amplification. In the analysis of the signal wave propagating structure, an expression will be obtained for the gain constant of the signal wave. The mismatch of the phase constants of pump line and signal line are then considered. Bandwidth and noise figure are briefly discussed. The effects of resistive power loss in the signal and pump lines are also considered. The importance of the relationship between the initial phase angles of pump, signal and idler waves is emphasized in the latter part of the analysis. Finally, an experimental traveling-wave parametric amplifier using magnetic thin film coated wire was constructed and operated to verify the results of the earlier analysis.

II. DESCRIPTION

A. General Power Relation

Manley and Rowe[3] have derived a very general set of equations relating power flowing into and out of an ideal nonlinear reactance. These relations are a powerful tool in predicting whether or not power gain is possible in a given situation, and in predicting the maximum gain that can be achieved.

The circuit model used by Manley and Rowe in their original derivation is a nonlinear capacitance. They derive the following two power relations independently.

$$\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{m \overline{W}_{mn}}{mf_o+nf_1} = 0 \qquad (1)$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{n \overline{W}_{mn}}{mf_o+nf_1} = 0 \qquad (2)$$

where

$\overline{W}_{mn}$ = the average power flowing into the nonlinear capacitor at various frequencies

f_o and f_1 = fundamental frequencies of the energies impressed across the capacitor

m and n = integers ranging from $-\infty$ to ∞

We shall obtain the above two equations simultaneously by using inductance as the circuit model and a simpler method of approach.

Consider a nonlinear inductor with a single-valued characteristic defined by

$$i = f(\lambda) \qquad (3)$$

where

i = the current in the inductor

λ = the flux linkage

i and λ may be expressed as double Fourier series

$$\lambda = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \lambda_{mn} \; e^{j(m\omega_o t + n\omega_1 t)} \tag{4}$$

$$i = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{mn} \; e^{j(m\omega_o t + n\omega_1 t)} \tag{5}$$

where

$$\omega_o = 2\pi f_o \quad , \quad \omega_1 = 2\pi f_1$$

f_o and f_1 are fundamental frequencies of the energies impressed across the inductor.

The voltage v across the inductor is

$$v = \frac{d\lambda}{dt} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} V_{mn} \; e^{j(m\omega_o t + n\omega_1 t)} \tag{6}$$

where

$$V_{mn} = j(m\omega_o + n\omega_1) \; \lambda_{mn} \tag{7}$$

The average power flowing into the nonlinear inductor at various frequencies is[19]

$$\begin{aligned} \overline{W}_{mn} &= 2 \; \mathrm{Re}[I_{mn} V^*_{mn}] \\ &= -2(m\omega_o + n\omega_1) \; \mathrm{Re}[jI_{mn}\lambda^*_{mn}] \\ &= -4\pi(mf_o + nf_1) \; \mathrm{Re}[jI_{mn}\lambda^*_{mn}] \end{aligned} \tag{8}$$

where Re denotes the real part of the expression and * denotes the complex conjugate of the quantity.

Assume the inductor is lossless, then by the conservation of energy, one has

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \overline{W}_{mn} = 0 \tag{9}$$

Equation (9) can be written as

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\overline{W}_{mn}}{mf_o+nf_1} (mf_o+nf_1) = 0 \tag{10}$$

since the summation is taken to include the power corresponding to each frequency only once. Therefore one can write Equation (10) as

$$f_o \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{m\overline{W}_{mn}}{mf_o+nf_1} + f_1 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{n\overline{W}_{mn}}{mf_o+nf_1} = 0 \tag{11}$$

From Equation (8) one knows that the quantities

$$\frac{\overline{W}_{mn}}{mf_o+nf_1}$$

are independent of the fundamental frequencies f_1 and f_o. Therefore, each term of Equation (11) must be separately equal to zero, i.e.

$$\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{m\,\overline{W}_{mn}}{mf_o+nf_1} = 0 \tag{12}$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{n\,\overline{W}_{mn}}{mf_o+nf_1} = 0 \tag{13}$$

Equations (12) and (13) are the general energy relations of Manley and Rowe. The original analysis of Manley and Rowe established each of the relations of (12) and (13) independently. Actually the general energy relations are obtained not from conservation of energy alone but depend also on the properties of the particular device. In the magnetic thin film traveling-wave parametric amplifier, it is the nonlinear inductance that causes new frequencies to arise.

For example, in the case of parametric amplification, we take

$f_o = f_p$ (pump frequency)

$f_1 = f_s$ (signal frequency)

The structure itself will only allow one lower sideband of f_p to propagate. This lower sideband of f_p is $f_i = f_p - f_s$ which is called the idler frequency. Thus from Equations (12) and (13), one gets

$$\frac{\overline{W}_{1,0}}{f_o} + \frac{\overline{W}_{1,-1}}{f_o - f_1} = 0 \qquad (14)$$

$$\frac{\overline{W}_{0,1}}{f_1} - \frac{\overline{W}_{1,-1}}{f_o - f_1} = 0 \qquad (15)$$

In terms of subscripts of pump (p), signal (s) and idler (i), Equations (14) and (15) can be written as

$$\frac{\overline{W}_p}{f_p} + \frac{\overline{W}_i}{f_i} = 0 \qquad (16)$$

$$\frac{\overline{W}_i}{f_i} - \frac{\overline{W}_s}{f_s} = 0 \qquad (17)$$

The pump energy (W_p) which flows into the inductor is always positive. Then, from Equation (16), W_i must be negative. By Equation (17), W_s is negative too. This indicates that the pump power flows into the nonlinear inductor while idler and signal powers flow out of it. The relations among these quantities are

$$\frac{\overline{W}_s}{f_s} = \frac{\overline{W}_i}{f_i} = \frac{\overline{W}_s+\overline{W}_i}{f_s+f_i} = \frac{\overline{W}_p}{f_p} \tag{18}$$

This frequency mixing relation must then be satisfied by any parametric device. From Equation (18), we know that signal power W_s can be amplified from a very small value to a value so large that it can be comparable to W_p. The signal gain of parametric device is due to this kind of energy transfer interaction.

B. Variable Inductance

The frequency mixing action of any parametric amplification depends on the variable reactance of the particular device. For this traveling-wave parametric amplifier, a magnetic thin film time-varying inductance is used as the variable reactance. An expression for the time-varying inductance will be derived.

The signal line (idler as well) inductance is obtained by winding the signal line around a thin magnetic film coated wire (pump line). The magnetic thin film is electro-deposited on 5 mil diameter beryllium-copper wire with the easy direction of magnetization circumferential to the axis of the wire and the hard direction parallel to the axis of the wire. The signal coil in which signal and idler currents flow is positioned in such a way that its fields of the coil is at right angle to the rest direction of the magnetization vector $\vec{M}$. The rest direction of the magnetization vector $\vec{M}$ is in the same direction as the bias field and the pump field. The physical arrangement of the signal coil and pump line is shown in Fig. 1.

When signal current flows in the coil it produces a field H_s which

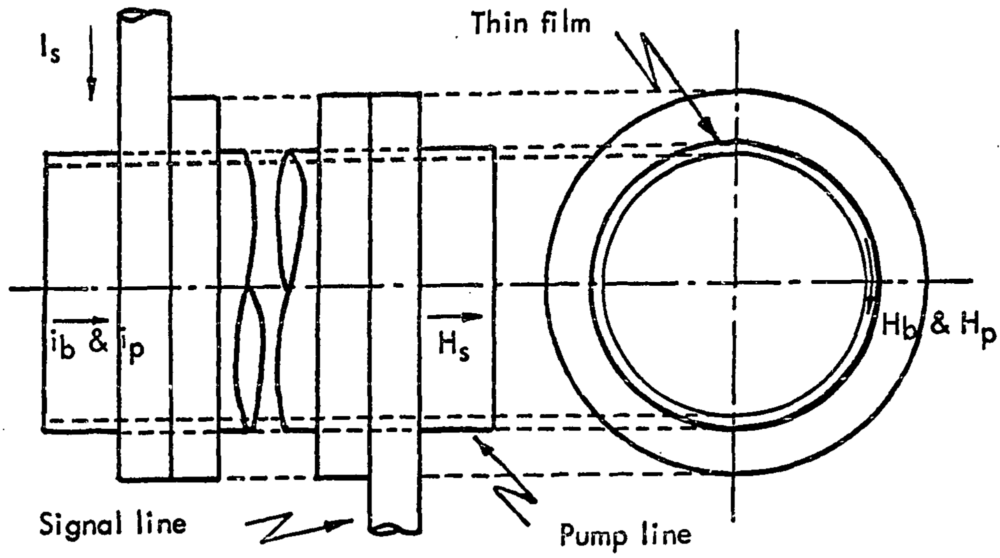

Fig. 1. Signal coil and pump line arrangement

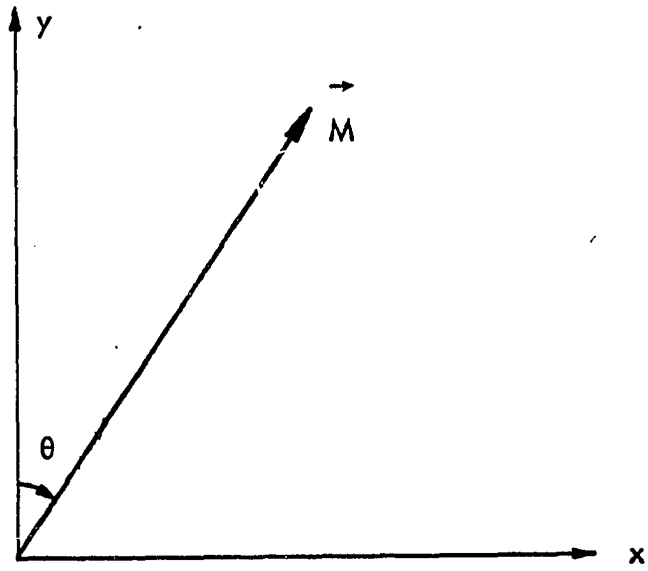

Fig. 2. Rectangular coordinate system for thin magnetic film

tends to rotate the magnetization vector $\vec{M}$ away from its rest direction (the same direction as H_p and H_b). We define the quiscent angle between $\vec{M}$ and the rest direction as θ. By considering the total energy of the magnetic thin film, one can determine the relationships between θ, H_s, H_b and H_p.

The total energy of the film is the sum of the magnetocrystalline anisotropy energy, the wall energy and the magnetostatic energy. The magnetocrystalline anisotropy energy is caused by the displacement of the magnetization vector $\vec{M}$ from a preferred crystal direction. This anisotropy energy is given by Equation (19).

$$\overline{W}_{an} = \frac{H_K M}{2} \sin^2\theta \tag{19}$$

where

$\overline{W}_{an}$ = anisotropy energy

M = magnetization vector

θ = angle between M and the preferred direction of M

H_K = anisotropy field

The wall energy is the energy due to the formation of domain walls. It is actually the sum of the exchange energy and the anisotropy energy of the wall. Since the thin film is considered to be a single domain, the wall energy will not be present.

The magnetostatic energy is given by the negative of the scalar product of the total magnetic field intensity and the magnetization vector. The first part of the magnetostatic energy is due to the interaction of magnetization vector with the externally applied fields. This applied field energy is

$$\overline{W}_{ap} = - \vec{H}_{ap} \cdot \vec{M} \tag{20}$$

where

$\overline{W}_{ap}$ = energy due to the applied fields

$\vec{H}_{ap}$ = applied fields

The second part of the magnetostatic energy is concerned with the return path for the flux of the magnetization vector. The magnetic field intensity of the return path flux, called the demagnetization field, is dependent upon the shape of the magnetic material. In this thesis the film geometry is cylindrical with the magnetization vector circumferentially directed so there is no demagnetizing field and therefore no demagnetizing energy.

Thus the total energy of the film is the sum of the anisotropy energy and the applied field energy.

$$\overline{W} = \frac{H_K M}{2} \sin^2\theta - \vec{H}_{ap} \cdot \vec{M} \tag{21}$$

where

W = total energy

Since the thickness of the magnetic thin film is very small compared to the radius of the pump line conductor, a rectangular coordinate system can be set up in the plane of the thin magnetic film with x-axis in the hard direction of the film and y-axis in the easy direction (see Figure 2). Then we can express the applied fields vector as

$$\vec{H}_{ap} = \hat{x} H_x + \hat{y} H_y \tag{22}$$

Then the dot product of the last term of Equation (21) becomes

$$\vec{H}_{ap} \cdot \vec{M} = M\,H_x \sin\theta + M\,H_y \cos\theta \tag{23}$$

Hence the total energy can be written as

$$\overline{W} = \frac{H_K M}{2} \sin^2\theta - H_x\,M \sin\theta - H_y\,M \cos\theta \tag{24}$$

The quiscent angle θ will be of such a value that the total energy of the film is minimum when the magnetization vector $\vec{M}$ is at the angle θ from its easy direction. Differentiate Equation (24) with respect to θ and set it to zero, one has

$$\frac{d\overline{W}}{d\theta} = H_K M \sin\theta \cos\theta - H_x M \cos\theta + H_y M \sin\theta = 0 \tag{25}$$

Solving the above equation for sinθ, one gets

$$\sin\theta = \frac{H_x}{H_K + \frac{H_y}{\cos\theta}} \tag{26}$$

Since H_x (signal field) is much smaller than H_K (anisotropy field) and H_y (pump and bias fields). Therefore θ is small and cosθ equals unity approximately. Thus Equation (26) becomes

$$\sin\theta = \frac{H_x}{H_K+H_y} = \frac{H_s}{H_K+(H_b+H_p)} \tag{27}$$

The flux density in the hard direction due to the magnetization vector $\vec{M}$ is

$$B_x = M \sin\theta \tag{28}$$

Substituting Equation (27) into (28), we have

$$B_x = M\,\frac{H_x}{H_K+H_y} \tag{29}$$

Thus the permeability of the film in the hard direction is

$$\mu_x = \frac{B_x}{H_x} = \frac{M}{H_K+H_y} \tag{30}$$

As mentioned before, the signal field H_s is in the hard direction. Hence the total flux along the hard direction is

$$\phi = \mu_o\, A\, H_s + \mu_x\, A_f\, H_s \tag{31}$$

For a signal coil of N turns, the flux linkage is

$$\lambda_s = N\, \mu_o\, A\, H_s + N\, \mu_x\, A_f\, H_s \tag{32}$$

where

N = number of turns per coil

μ_x = permeability of the magnetic material

μ_o = permeability of vacuum

H_s = signal field intensity

A = cross section area of the signal coil

A_f = cross section area of the magnetic thin film

H_s is related to the signal current by

$$H_s = N\, I_s \tag{33}$$

Substituting Equations (33) and (30) into Equation (32), one has

$$\lambda_s = N^2\, (\mu_o\, A + \frac{M\, A_f}{H_K+H_y})\, I_s \tag{34}$$

Thus one gets the signal line inductance as

$$L_s = \frac{\lambda_s}{I_s} = N^2\, (\mu_o\, A + \frac{M\, A_f}{H_K+H_y}) \tag{35}$$

From the above equation, one can see that the signal inductance come from two sources. The first term is for air inductance and the second term is due to the contribution of the thin film.

H_y represents the field in the easy direction which consists of the time-varying pump field H_p and the d.c. bias field H_b. Thus one has

$$H_y = H_b + H_p \cos \omega_p t \tag{36}$$

where

$$\omega_p = \text{pump frequency}$$

Substituting Equation (36) into (35), one gets

$$L_s = N^2 \mu_o A + \frac{N^2 M A_f}{H_K + H_b + H_p \cos \omega_p t}$$

$$= L_a + L_f \left(\frac{1}{1 + a \cos \omega_p t}\right) \tag{37}$$

where

$$L_a = N^2 \mu_o A \tag{38}$$

$$L_f = \frac{N^2 M A_f}{H_K + H_b} \tag{39}$$

$$a = \frac{H_p}{H_K + H_b} \tag{40}$$

Equation (37) can be expanded as

$$L_s = L_a + L_f[1 - a \cos \omega_p t + (a \cos \omega_p t)^2 - \ldots.] \tag{41}$$

Since H_p is much smaller than (H_K+H_b), a must be small. Besides higher frequency modes are not allowed to propagate in the structure. Therefore one can neglect the higher order terms of Equation (41). L_s can then be written as

$$L_s = L_a + L_f(1-a \cos \omega_p t)$$

$$= L_1 + L e^{j\omega_p t}$$

$$= L_1(1+\eta e^{j\omega_p t}) \tag{42}$$

where

$$L_1 = L_a + L_f$$
$$L = a L_f \tag{43}$$
$$\eta = \frac{L}{L_1}$$

L_1 is the static signal line inductance which will be used in the following analysis. L stands for the magnitude of the time-varying inductance. Actually in the analysis of the traveling-wave propagating structure, the energy coupling inductance L(z,t) is not only time-varying but also a function of position z. One may write L(z,t) as

$$L(z,t) = \frac{1}{2} L[e^{j(\omega_p t-\beta_p z)} + e^{-j(\omega_p t-\beta_p z)}] \tag{44}$$

where

ω_p = pump frequency

β_p = pump line phase constant

III. ANALYSES

A. Analysis of the Propagating Structure

The purpose of this Chapter is to investigate the properties of a time-varying distributed reactance in a propagating structure. We shall analyze two propagating waves of frequencies, ω_1 and ω_2. The waves are coupled through a distributed reactance which varies in time at a frequency ω_p, and in distance by a phase constant β_p. Here ω_p and β_p are known as the pump frequency and the pump phase constant. The pump frequency is the sum of ω_1 and ω_2. For simplification we shall assume waves of frequencies other than ω_1, ω_2, and ω_p to be suppressed in the propagating structure. This can be done by properly designing a propagating structure such that the other side bands are outside the pass band of the circuit.

Assume for convenience that the ω_1 and ω_2 waves are carried by two propagating circuits namely signal and idler circuits. The two circuits are coupled through a distributed inductance L(z,t) which varies with time t and distance z. i.e.

$$L(z,t) = \frac{1}{2}[L(z)\ e^{j\omega_p t} + L^*(z)\ e^{-j\omega_p t}]$$

$$= \frac{1}{2}\ L[e^{j(\omega_p t-\beta_p z)} + e^{-j(\omega_p t-\beta_p z)}] \tag{45}$$

As described in the previous Chapter, the variable inductance may be obtained by feeding an electromagnetic wave (generally known as the pumping wave) to a thin film coated pump line such that the equivalent inductance of the signal or idler circuit varies with the intensity of the pumping field. Equation (45) denotes a forward traveling wave for positive β_p. The pumping wave is supplied by a local oscillator of frequency ω_p and has a power level substantially larger than that of the ω_1 and ω_2 waves. $L(z,t)$ is the coupling element between pump power and signal (or idler) power. We should therefore bear in mind that $L(z,t)$ is not a passive element. It may absorb or supply power.

Dividing the propagating circuits into small sections, we may represent each section of the circuits by a filter type network as shown in Fig. 3. For simplification, the ohmic loss in the circuits is assumed to be small and may be neglected at first. Let the signal line have a series inductance L_1 and a shunt capacitance C_1, and let the idler line have a series inductance L_2 and a shunt capacitance C_2. The phase constants and characteristic impedances of the signal line and idler line are respectively,

$$\begin{aligned} \beta_1 &= \omega_1 \sqrt{L_1 C_1} \quad , \quad Z_{01} = \sqrt{\frac{L_1}{C_1}} \\ \beta_2 &= \omega_2 \sqrt{L_2 C_2} \quad , \quad Z_{02} = \sqrt{\frac{L_2}{C_2}} \end{aligned} \tag{46}$$

Let $V_1(z,t)$, $V_2(z,t)$, $I_1(z,t)$ and $I_2(z,t)$ be the voltages and currents of signal and idler circuits. Then we have,

$$\frac{\partial V_1(z,t)}{\partial z} + L_1 \frac{\partial I_1(z,t)}{\partial t} + \frac{\partial}{\partial t}\left[L(z,t) I_2(z,t)\right] = 0 \tag{47}$$

$$\frac{\partial I_1(z,t)}{\partial z} + C_1 \frac{\partial V_1(z,t)}{\partial t} = 0 \tag{48}$$

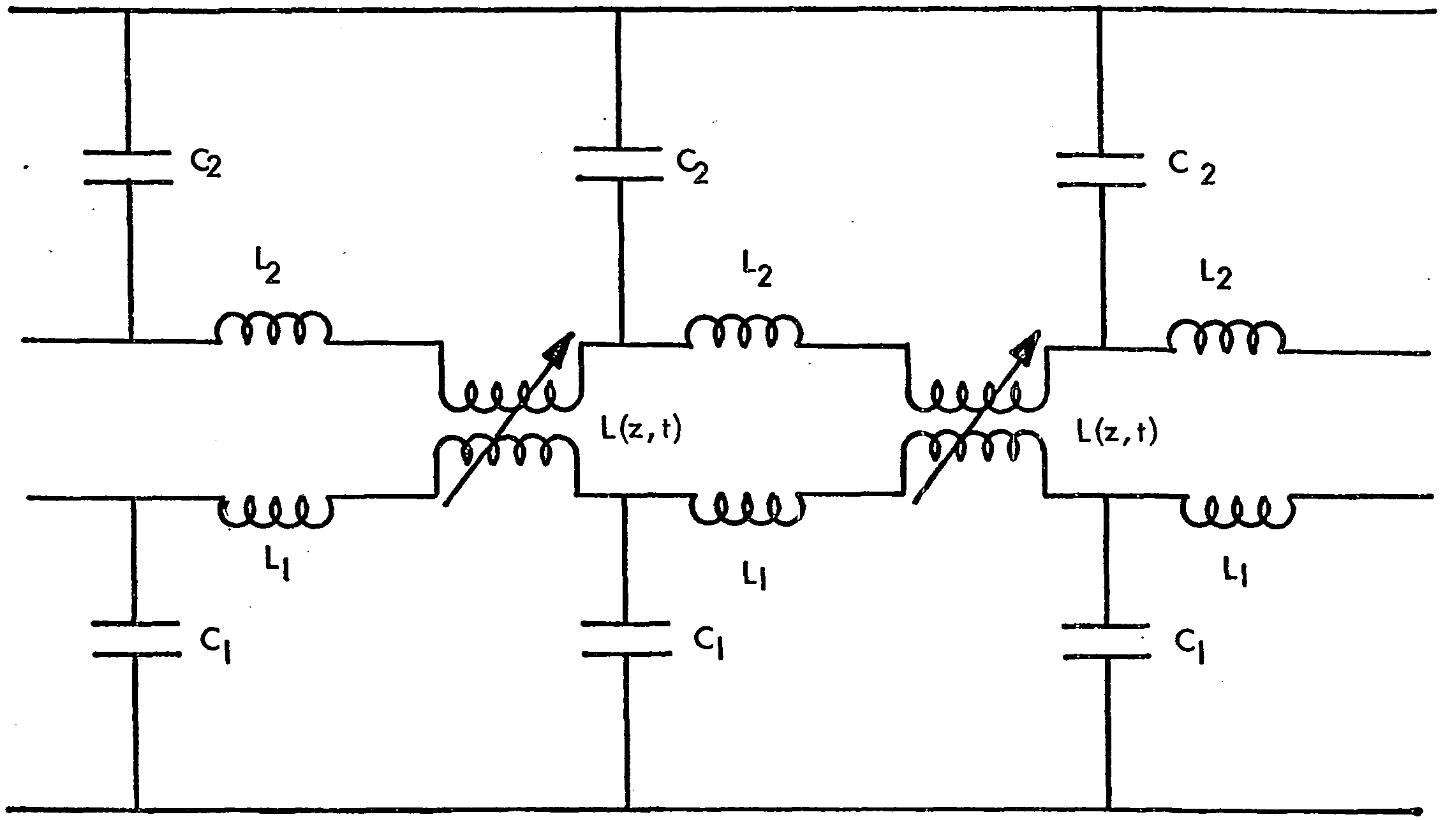

Fig. 3. Signal and idler wave propagating structure

$$\frac{\partial V_2(z,t)}{\partial z} + L_2 \frac{\partial I_2(z,t)}{\partial t} + \frac{\partial}{\partial t} [L(z,t)I_1(z,t)] = 0 \tag{49}$$

$$\frac{\partial I_2(z,t)}{\partial z} + C_2 \frac{\partial V_2(z,t)}{\partial t} = 0 \tag{50}$$

Differentiate (48) with respect to z and differentiate (47) with respect to t.

$$\frac{\partial^2 I_1(z,t)}{\partial z^2} = - C_1 \frac{\partial^2 V_1(z,t)}{\partial t \partial z} \tag{51}$$

$$\frac{\partial^2 V_1(z,t)}{\partial t \partial z} = - L_1 \frac{\partial^2 I_1(z,t)}{\partial t^2} = \frac{\partial^2}{\partial t^2} [L(z,t)I_2(z,t)] \tag{52}$$

Substituting the value of $\frac{\partial^2 V_1(z,t)}{\partial t \partial z}$ in (52) into Equation (51) we have

$$\frac{\partial^2 I_1(z,t)}{\partial z^2} = C_1 L_1 \frac{\partial^2 I_1(z,t)}{\partial t^2} + C_1 \frac{\partial^2}{\partial t^2} [L(z,t)I_2(z,t)] \tag{53}$$

Similarly from Equations (49) and (50) we get

$$\frac{\partial^2 I_2(z,t)}{\partial z^2} = C_2 L_2 \frac{\partial^2 I_2(z,t)}{\partial t^2} + C_2 \frac{\partial^2}{\partial t^2} [L(z,t)I_1(z,t)] \tag{54}$$

For parametric amplification, we assume the frequency ω_p and phase constant β_p of the pump wave be such that

$$\begin{aligned} \omega_p &= \omega_1 + \omega_2 \\ \beta_p &= \beta_1 + \beta_2 \end{aligned} \tag{55}$$

now put

$$I_1(z,t) = \frac{1}{2}[I_1(z)e^{j\omega_1 t} + I_1^*(z)e^{-j\omega_1 t}]$$
$$I_2(z,t) = \frac{1}{2}[I_2(z)e^{j\omega_2 t} + I_2^*(z)e^{-j\omega_2 t}] \tag{56}$$

then the last term of Equation (53) becomes

$$C_1 \frac{\partial^2}{\partial t^2}[L(z,t)I_2(z,t)]$$

$$= \frac{C_1}{4}\frac{\partial^2}{\partial t^2}[L(z)e^{j\omega_p t} + L^*(z)e^{-j\omega_p t}][I_2(z)e^{j\omega_2 t} + I_2^*(z)e^{-j\omega_2 t}]$$

$$= \frac{C_1}{4}\frac{\partial^2}{\partial t^2}\{L(z)I_2(z)e^{j(\omega_p+\omega_2)t} + L^*(z)I_2^*(z)e^{-j(\omega_p+\omega_2)t}$$
$$+ L(z)I_2^*(z)e^{j(\omega_p-\omega_2)t} + L^*(z)I_2(z)e^{-j(\omega_p-\omega_2)t}\}$$

$$= \frac{C_1}{4}\frac{\partial^2}{\partial t^2}\{L(z)I_2^*(z)e^{j\omega_1 t} + L^*(z)I_2(z)e^{-j\omega_1 t}\}$$

$$= -\frac{C_1\omega_1^2}{4}[L(z)I_2^*(z)e^{j\omega_1 t} + L^*(z)I_2(z)e^{-j\omega_1 t}] \tag{57}$$

Similarly, the last term of Equation (54) is

$$C_2 \frac{\partial^2}{\partial t^2}[L(z,t)I_1(z,t)]$$

$$= \frac{C_2}{4}\frac{\partial^2}{\partial t^2}[L(z)I_1^*(z)e^{j\omega_2 t} + L^*(z)I_1(z)e^{-j\omega_2 t}]$$

$$= -\frac{C_2\omega_2^2}{4}[L(z)I_1^*(z)e^{j\omega_2 t} + L^*(z)I_1(z)e^{-j\omega_2 t}] \tag{58}$$

By substituting Equations (56) and (57) into Equation (53) we get the following two equations:

$$\frac{\partial^2 I_1(z)}{\partial z^2} = - \omega_1^2 L_1 C_1 I_1(z) - \frac{1}{2} \omega_1^2 C_1 L(z) I_2^*(z) \tag{59}$$

$$\frac{\partial^2 I_1^*(z)}{\partial z^2} = - \omega_1^2 L_1 C_1 I_1^*(z) - \frac{1}{2} \omega_1^2 C_1 L^*(z) I_2(z) \tag{60}$$

Likewise substituting Equations (56) and (58) into Equation (54) we get,

$$\frac{\partial^2 I_2(z)}{\partial z^2} = - \omega_2^2 L_2 C_2 I_2(z) - \frac{1}{2} \omega_2^2 C_2 L(z) I_1^*(z) \tag{61}$$

$$\frac{\partial^2 I_2^*(z)}{\partial z^2} = - \omega_2^2 L_2 C_2 I_2^*(z) - \frac{1}{2} \omega_2^2 C_2 L^*(z) I_1(z) \tag{62}$$

Now we can see that Equations (59) and (62) are a coupled pair, while Equations (60) and (61) are another coupled pair. We shall consider the coupled pair of Equations (59) and (62) only.

$$\begin{aligned} I_1(z) &= A_1(z)e^{-j\beta_1 z} \quad , \quad I_1^*(z) = A_1^*(z)e^{j\beta_1 z} \\ I_2(z) &= A_2(z)e^{-j\beta_1 z} \quad , \quad I_2^*(z) = A_2^*(z)e^{j\beta_2 z} \\ L(z) &= L\ e^{-j\beta z} \quad , \quad L^*(z) = L\ e^{j\beta z} \end{aligned} \tag{63}$$

Substituting (63) into Equations (59) and (62) and rearranging terms give the following two equations

$$\frac{\partial^2 A_1(z)}{\partial z^2} - \beta_1^2 A_1(z) - 2j\beta_1 \frac{\partial A(z)}{\partial z} + \omega_1^2 L_1 C_1 A_1(z) + \frac{1}{2} \omega_1^2 C_1 L A_2^*(z) = 0 \tag{64}$$

$$\frac{\partial^2 A_2^*(z)}{\partial z^2} - \beta_2{}^2 A_2^*(z) + 2j\beta_2 \frac{\partial A_2^*(z)}{\partial z} + \omega_2{}^2 L_2 C_2 A_2^*(z)$$

$$+ \frac{1}{2}\,\omega_2{}^2 C_2 L A_1(z) = 0 \qquad (65)$$

From Equation (43) we know that L is much smaller than L_1. The coupling coefficients in Equations (47) and (49) will be assumed to be constant with z because the length over which the coupling occurs is much much less than a wavelength at the pump frequency. $A_1(z)$ and $A_2^*(z)$ are functionally related to the coupling coefficient, hence the terms

$\frac{\partial^2 A_1(z)}{\partial z^2}$ and $\frac{\partial^2 A_2^*(z)}{\partial z^2}$ can be neglected. Besides, from (46)

$$\beta_1{}^2 = \omega_1{}^2 L_1 C_1$$

$$\beta_2{}^2 = \omega_2{}^2 L_2 C_2$$

Thus the second and fourth terms of (64) and (65) are cancelled out. Then (64) and (65) can be reduced to

$$\frac{\partial A_1(z)}{\partial z} = \frac{\omega_1{}^2 C_1 L}{4j\beta_1} A_2^*(z) \qquad (66)$$

$$\frac{\partial A_2^*(z)}{\partial z} = \frac{-\omega_2{}^2 C_2 L}{4j\beta_2} A_1(z) \qquad (67)$$

Differentiate (61) with respect to z, we get

$$\frac{\partial^2 A_1(z)}{\partial z^2} = \frac{\omega_1{}^2 C_1 L}{4j\beta_1} \frac{\partial A_2^*(z)}{\partial z} \qquad (68)$$

Combining (67) and (68), we then have

$$\frac{\partial^2 A_1(z)}{\partial z^2} = \frac{\omega_1^{\,2}\omega_2^{\,2} C_1 C_2 L^2}{16\beta_1\beta_2} A_1(z) \tag{69}$$

The solution of Equation (69) is of the form

$$A_1(z) = a_1 e^{\alpha z} + a_2 e^{-\alpha z} \tag{70}$$

where

$$\alpha = \text{gain constant}$$

$$= \left(\frac{\omega_1^{\,2}\omega_2^{\,2} C_1 C_2 L^2}{16\beta_1\beta_2}\right)^{\frac{1}{2}} \tag{71}$$

By Equation (46), the gain constant α can be written as

$$\alpha = \frac{L}{4}\sqrt{\frac{\omega_1\omega_2}{z_{o1} z_{o2}}} \tag{72}$$

Substituting Equation (70) into Equation (66), we get

$$A_2^*(z) = \frac{4j\beta_1}{\omega_1^{\,2} C_1 L}\frac{\partial A_1(z)}{\partial z}$$

$$= \frac{4j\beta_1}{\omega_1^{\,2} C_1 L}[a_1\alpha e^{\alpha z} - a_2\alpha e^{-\alpha z}] \tag{73}$$

Then the signal and idler current can be written as

$$I_1(z,t) = 2\mathrm{Re}[A_1(z)e^{j(\omega_1 t - \beta_1 z)}]$$

$$= 2\mathrm{Re}[(a_1 e^{\alpha z} + a_2 e^{-\alpha z})e^{j(\omega_1 t - \beta_1 z)}] \tag{74}$$

$$I_2(z,t) = 2\mathrm{Re}[A_2^*(z)e^{j(\omega_2 t - \beta_2 z)}]$$

$$= 2\mathrm{Re}[\frac{4j\beta_1}{\omega_1^{\,2} C_1 L}(a_1\alpha e^{\alpha z} - a_2\alpha e^{-\alpha z})e^{j(\omega_2 t - \beta_2 z)}] \tag{75}$$

The boundary conditions at z=0 are

$$I_1(0,t) = A \cos(\omega_1 t + \sigma) \tag{76}$$

$$I_2(0,t) = 0 \tag{77}$$

By (74) and (76), we have

$$2\mathrm{Re}[(a_1+a_2)e^{j\omega_1 t}] = A \cos(\omega_1 t + \sigma) \tag{78}$$

From (75) and (77) we get

$$a_1 = a_2 \tag{79}$$

Substituting (79) into (78) we find the condition

$$4\mathrm{Re}[a_1 e^{j\omega_1 t}] = A \cos(\omega_1 t + \sigma)$$

this indicates that

$$a_1 = a_2 = \frac{A}{4} e^{j\sigma}$$

Then the final solution of signal and idler currents are

$$I_1(z,t) = 2\mathrm{Re}[\frac{A}{4} e^{j\sigma} (e^{\alpha z} + e^{-\alpha z}) e^{j(\omega_1 t - \beta_1 z)}]$$

$$= \frac{A}{2}(e^{\alpha z} + e^{-\alpha z}) \cos(\omega_1 t - \beta_1 z + \sigma) \tag{80}$$

$$I_2(z,t) = 2\mathrm{Re}[\frac{j4\beta_1}{\omega_1^2 C_1 L} \alpha \frac{A}{4} e^{j\sigma}(e^{\alpha z} - e^{-\alpha z}) e^{j(\omega_1 t - \beta\ z)}$$

$$= \frac{-A}{2}\sqrt{\frac{L_1\beta_2}{L_2\beta_1}}\,(e^{\alpha z} - e^{-\alpha z}) \sin(\omega_1 t - \beta_1 z + \sigma) \tag{81}$$

Now we try to prove that the power relation of the signal and idler lines agrees with the Manley and Rowe equation. We know that the power carried by the signal line in frequency ω_1 is

$$P_1(z) = [I_1^{\,2}(z,t)\ z_{o1}]_{ave.} \tag{82}$$

By neglecting the decaying wave of Equation (80) we take

$$I_1(z,t) = \frac{A}{2} e^{\alpha z} \cos\ (\omega_1 t - \beta_1 z + \sigma) \tag{83}$$

then

$$I^2(z,t)]_{ave.} = \frac{1}{4} A^2 e^{2\alpha z}\ \overline{\cos^2(\omega_1 t - \beta_1 z + \sigma)}$$

$$= \frac{1}{8} A^2 e^{2\alpha z}$$

thus

$$P_1(z) = \frac{1}{8} A^2 e^{2\alpha z} \sqrt{\frac{L_1}{C_1}} \tag{84}$$

Similarly by neglecting the decaying wave of Equation (81), the power carried by idler line in frequency ω_2 is

$$P_2(z) = [I_2^{\,2}(z,t)\ z_{02}]_{ave.}$$

$$= \frac{1}{8} A^2 \frac{L_1 \beta_2}{L_2 \beta_1} e^{2\alpha z} \sqrt{\frac{L_2}{C_2}}$$

$$= \frac{1}{8} A^2 \frac{\omega_2}{\omega_1} e^{2\alpha z} \sqrt{\frac{L_1}{C_1}} \tag{85}$$

from (84) and (85) we then get

$$\frac{P_1(z)}{\omega_1} = \frac{P_2(z)}{\omega_2} = \frac{P_1(z)+P_2(z)}{\omega_1+\omega_2} \tag{86}$$

This is the Manley and Rowe relation.

Now we consider the case where the pump line and signal line (idler line as well) structures are not ideally matched. That is in the case of $\beta \neq \beta_1+\beta_2$.

Put $\beta = \beta_1{}'+\beta_2{}'$

$$= \beta_1+\beta_2+\Delta\beta \tag{87}$$

where

$$\beta_1{}' = \beta_1+k_1\Delta\beta \tag{88}$$

$$\beta_2{}' = \beta_2+k_2\Delta\beta \tag{89}$$

$$k_1+k_2 = 1 \tag{90}$$

Then we may write

$$I_1(z) = A_1(z)e^{-j\beta_1{}'t} = A_1(z)e^{-j(\beta_1+k_1\Delta\beta)z}$$

$$I_1^*(z) = A_1^*(z)e^{j\beta_1{}'z} = A_1^*(z)e^{j(\beta_1+k_1\Delta\beta)z}$$

$$I_2(z) = A_2(z)e^{-j\beta_2{}'z} = A_2(z)e^{-j(\beta_2+k_2\Delta\beta)z}$$

$$I_2^*(z) = A_2^*(z)e^{j\beta_2{}'z} = A_2^*(z)e^{j(\beta_2+k_2\Delta\beta)z} \tag{91}$$

As before, we consider the pair $A_1(z)$ and $A_2^*(z)$ only. The corresponding equations of (64) and (65) for this more general case are

$$\frac{\partial^2 A_1(z)}{\partial z^2} - \beta_1{}'^2 A_1(z) - 2j\beta_1{}' \frac{\partial A_1(z)}{\partial z} + \omega_1{}^2 L_1 C_1 A_1(z)$$

$$+ \frac{1}{2}\,\omega_1{}^2 C_1 L A_2^*(z) = 0 \tag{92}$$

$$\frac{\partial^2 A_2(z)}{\partial z^2} - \beta_2'^2 A_2^*(z) + 2j\beta_2' \frac{\partial A_2^*(z)}{\partial z} + \omega_2^2 L_2 C_2 A_2^*(z)$$

$$+ \frac{1}{2}\omega_2^2 C_2 L A_1(z) = 0 \qquad (93)$$

By substituting (88) and (89) into (92) and (93), we get

$$\frac{\partial^2 A_1(z)}{\partial z^2} - (\beta_1^2 + 2\beta_1 k_1 \Delta\beta + k_1^2 \Delta\beta^2) A_1(z) - 2j(\beta_1 + k_1 \Delta\beta) \frac{\partial A_1(z)}{\partial z}$$

$$+\omega_1^2 L_1 C_1 A_1(z) + \frac{1}{2}\omega_1^2 C_1 L A_2^*(z) = 0 \qquad (94)$$

$$\frac{\partial^2 A_2^*(z)}{\partial z^2} - (\beta_2^2 + 2\beta_2 k_2 \Delta\beta + k_2^2 \Delta\beta^2) A_2^*(z) + 2j(\beta_2 + k_2 \Delta\beta) \frac{\partial A_2^*(z)}{\partial z}$$

$$+\omega_2^2 L_2 C_2 A_2^*(z) + \frac{1}{2}\omega_2^2 C_2 L A_1(z) = 0 \qquad (95)$$

As before we neglect the higher order terms $\frac{\partial^2 A_1(z)}{\partial z^2}$ and $\frac{\partial^2 A_2^*(z)}{\partial z^2}$.

The terms $k_1^2 \Delta\beta^2$ and $k_2^2 \Delta\beta^2$ are also neglected because they are much smaller than $2\beta_1 k_1 \Delta\beta$ and $2\beta_2 k_2 \Delta\beta$. In addition, the terms $\beta_1^2 A_1(z)$, $\beta_2^2 A_2^*(z)$ and terms $\omega_1^2 L_1 C_1 A_1(z)$, $\omega_2^2 L_2 C_2 A_2^*(z)$ cancel out. Therefore Equations (94) and (95) can be reduced to

$$\frac{\partial A_1(z)}{\partial z} - jk_1 \Delta\beta A_1(z) = -j \frac{1}{4\beta_1} \omega_1^2 C_1 L A_2^*(z) \qquad (96)$$

$$\frac{\partial A_2^*(z)}{\partial z} + jk_2 \Delta\beta A_2^*(z) = j \frac{1}{4\beta_2} \omega_2^2 C_2 L A_1(z) \qquad (97)$$

We now can see the importance of $\Delta\beta$. If $k_1\Delta\beta$ and $k_2\Delta\beta$ are much larger than $\frac{\omega_1{}^2 C_1 L}{4\beta_1}$ and $\frac{\omega_2{}^2 C_2 L}{4\beta_2}$, we can neglect the last terms of Equations (96) and (97). That means the coupling effect due to the variable driving inductance L vanishes. Therefore in order to obtain the growing wave effect, we must keep $\Delta\beta$ as small as possible. Let us now investigate the case where these terms have comparable magnitudes. Equations (96) and (97) can be written as

$$\begin{cases} [\frac{\partial}{\partial z} - jk_1\Delta\beta]A_1(z) + j\frac{\omega_1{}^2 C_1 L}{4\beta_1} A_2^*(z) = 0 \\ -j\frac{\omega_2{}^2 C_2 L}{4\beta_2} A_1(z) + [\frac{\partial}{\partial z} + jk_2\Delta\beta]A_2^*(z) = 0 \end{cases} \quad (98)$$

Combine the above two simultaneous differential equations into a single ordinary differential equation with dependent variable $A_1(z)$. We have

$$\frac{d^2A_1(z)}{dz^2} + j(k_2-k_1)\Delta\beta\,\frac{dA_1(z)}{dz} + (k_1k_2\Delta\beta^2 - \frac{\omega_1{}^2\omega_2{}^2 C_1 C_2 L^2}{16\beta_1\beta_2})\,A_1(z) = 0 \quad (99)$$

In the actual construction of the traveling-wave structure, the signal and idler waves are traveling in the same element. Thus, for our degenerate case we have

$$k_1 = k_2 = k \quad (100)$$

$$C_1 = C_2 \quad , \quad L_1 = L_2 \quad (101)$$

Therefore Equation (99) reduces to

$$\frac{d^2A_1(z)}{dz^2} = [\frac{\omega_1{}^2\omega_2{}^2 C_1{}^2 L^2}{16\beta_1\beta_2} - k^2\Delta\beta^2]A(z) \quad (102)$$

since

$$k_1 + k_2 = 1 .$$

thus

$$k = k_1 = k_2 = \frac{1}{2}$$

if

$$\left|\frac{1}{2}\Delta\beta\right| < \left|\frac{\omega_1\omega_2 C_1 L}{4\sqrt{\beta_1\beta_2}}\right|$$

Then solution of $A_1(z)$ is of the form

$$A_1(z) = a_1 e^{\alpha' z} + a_2 e^{-\alpha' z} \tag{103}$$

where

$$\alpha' = \left[\frac{\omega_1^2\omega_2^2 C_1^2 L^2}{16\beta_1\beta_2} - \left(\frac{1}{2}\Delta\beta\right)^2\right]^{\frac{1}{2}}$$

With the same boundary conditions as in (76) and (77), the final solutions of signal and idler currents are in the same forms as in Equations (80) and (81) using α' and β' instead of α and β. Equation (103) still represents the existence of a growing wave, but the gain constant of the structure is reduced due to $\Delta\beta$.

If

$$\left|\frac{1}{2}\Delta\beta\right| > \left|\frac{\omega_1\omega_2 C_1 L}{4\sqrt{\beta_1\beta_2}}\right|$$

then the solution of $A_1(z)$ is of the form

$$A_1(z) = a_1 e^{j\alpha' z} + a_2 e^{-j\alpha' z} \tag{104}$$

where

$$\alpha' = [(\tfrac{1}{2}\Delta\beta)^2 - \frac{\omega_1{}^2\omega_2{}^2 D_1{}^2 L^2}{16\beta_1\beta_2}]^{\frac{1}{2}}$$

From Equation (96), $A_2^*(z)$ can be found as

$$A_2^*(z) = \frac{4j\beta_1}{\omega_1{}^2 C_1 L}[\frac{dA_1(z)}{dz} - jk_1\Delta\beta A_1(z)]$$

$$= \frac{4\ \beta_1}{\omega_1{}^2 C_1 L}\{[\frac{\Delta\beta}{2} - ((\frac{\Delta\beta}{2})^2 - \zeta^2)^{\frac{1}{2}}]a_1 e^{j\frac{1}{2}[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}}z}$$

$$+ a_2[\frac{\Delta\beta}{2} + ((\frac{\Delta\beta}{2})^2 - \zeta^2)^{\frac{1}{2}}]e^{-j\frac{1}{2}[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}}z}\} \qquad (105)$$

where

$$\zeta^2 = \frac{\omega_1{}^2\omega_2{}^2 C_1{}^2 L^2}{16\beta_1\beta_2} \qquad (106)$$

As in Equations (74) and (75), the signal and idler currents can be written as

$$I_1(z,t) = 2\text{Re}[A_1(z)e^{j(\omega_1 t - \beta_1 z)}]$$

$$= 2\text{Re}\{[a_1 e^{j((\frac{\Delta\beta}{2})^2 - \zeta^2)^{\frac{1}{2}}z} + a_2 e^{-j((\frac{\Delta\beta}{2})^2 - \zeta^2)^{\frac{1}{2}}z}]e^{j(\omega_1 t - \beta_1 z)}\}$$

$$(107)$$

$$I_2(z,t) = 2\text{Re}[A_2^*(z)e^{j(\omega_2 t - \beta_2 z)}]$$

$$= 2\text{Re}\{\frac{2\beta_1}{\omega_1{}^2 C_1 L}[a_1(\frac{\Delta\beta}{2} - ((\frac{\Delta\beta}{2})^2 - \zeta^2)^{\frac{1}{2}})e^{j\ ((\frac{\Delta\beta}{2})^2 - \zeta^2)^{\frac{1}{2}}z}$$

$$+ a_2[\frac{\Delta\beta}{2} + ((\frac{\Delta\beta}{2})^2 - \zeta^2)^{\frac{1}{2}})e^{-j\ [[\frac{\Delta\beta}{2}]^2 - \zeta^2]^{\frac{1}{2}}z}]e^{j(\omega_2 t - \beta_2 z)}\} \qquad (108)$$

Suppose the boundary conditions are

$$I_1(0,t) = I_{10} \cos(\omega_1 t + \sigma) \tag{109}$$

$$I_2(0,t) = 0 \tag{110}$$

From Equations (107) and (109) we have

$$2\mathrm{Re}\{[a_1+a_2]e^{j\omega_1 t}\} = I_{10} \cos(\omega_1 t + \sigma) \tag{111}$$

From Equations (108) and (110) we have

$$\left[\frac{\Delta\beta}{2} - \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}\right]a_1 + \left[\frac{\Delta\beta}{2} + \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}\right]a_2 = 0 \tag{112}$$

Solving Equations (111) and (112) we get

$$a_1 = \frac{I_{10}e^{j\sigma}\left[\frac{\Delta\beta}{2} + \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}\right]}{4\left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}} \tag{113}$$

$$a_2 = \frac{-I_{10}e^{j\sigma}\left[\frac{\Delta\beta}{2} - \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}\right]}{4\left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}} \tag{114}$$

Substituting (113) and (114) into (107) and (108) we have

$$I_1(z,t) = \mathrm{Re}\left\{\frac{\zeta^2 I_{10}}{2\left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}}\left[\frac{e^{j\left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}} z}}{\frac{\Delta\beta}{2} - \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}}\right.\right.$$

$$\left.\left. - \frac{e^{-j\left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}} z}}{\frac{\Delta\beta}{2} + \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}}\right]e^{j(\omega_1 t - \beta_1' z + \sigma)}\right\} \tag{115}$$

$$I_2(z,t) = \mathrm{Re}\left[\frac{j\zeta^2 \beta_1' I_{10}}{\omega_1{}^2 C_1 L} \left[\sin \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}} z\right] e^{j(\omega_2 t - \beta_2 z + \sigma)}\right] \tag{116}$$

Another form of writing signal and idler currents is

$$I_1(z,t) = \frac{\zeta^2 I_{10}}{2\left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}}$$

$$\left[\frac{\cos \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}} z \cos (\omega_1 t - \beta_1' z + \sigma) - \sin \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}} z \sin(\omega_1 t - \beta_1' z + \sigma)}{\frac{\Delta\beta}{2} - \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}}\right.$$

$$\left. - \frac{\cos\left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}} z \cos(\omega_1 t - \beta_1' z + \sigma) + \sin\left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}} z \sin(\omega_1 t - \beta_1' z + \sigma)}{\frac{\Delta\beta}{2} + \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{11}{2}}}\right] \tag{117}$$

$$I_2(z,t) = \frac{-\zeta^2 \beta_1' I_{10}}{\omega_1{}^2 C_1 L} \sin \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}} z \sin(\omega_2 t - \beta_2' z + \sigma) \tag{118}$$

From Equations (117) and (118), we see that the signal and idler currents are sine and cosine functions of z. This if $\Delta\beta$ is larger than the gain factor ζ, the gain of the structure varies periodically instead of growing exponentially. Nevertheless, it still has considerable gain due to the factor

$$\frac{\zeta^2}{2\left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}\left[\frac{\Delta\beta}{2} - \left(\left(\frac{\Delta\beta}{2}\right)^2 - \zeta^2\right)^{\frac{1}{2}}\right]} .$$

B. Bandwidth

A number of workers[6,8,10,11] have discussed versions of parametric amplifiers which use a single variable element and one or more resonant circuits. This type of parametric amplifier gives high gain but is extremely narrow band. It is possible to distribute the variable elements in a nonresonant, propagating circuit and obtain the advantages of greatly increased bandwidth and more stable operation.

Our traveling-wave parametric amplifier is intended to obtain the advantage of wide bandwidth. In order to get the parametric mixing effect, we require that

$$\omega = \omega_1 + \omega_2 \tag{119}$$

$$\beta = \beta_1 + \beta_2 \tag{120}$$

ordinarily

$$\beta = \beta_1 + \beta_2 + \Delta\beta \tag{121}$$

For the most efficient operation $\Delta\beta$ should be equal to zero. The pump frequency ω and the pump phase constant β are assumed to be fixed. In order for the condition of (119) to be met, a change in frequency from ω_1 to $(\omega_1+\Delta\omega)$ requires a similar change in ω_2. Similarly a change of phase constant from β_1 to $(\beta_1+\Delta\beta)$ requires a similar change in β_2. Thus, for broad-band operation, we require

$$\left(\frac{d\omega}{d\beta}\right)_1 = \left(\frac{d\omega}{d\beta}\right)_2 \tag{122}$$

that is, the group velocities of the signal and idler circuits must be equal. Since we use the same propagating structure for both the signal and idler waves, a change in frequency $\Delta\omega$ will produce some change, $\Delta\beta$, in the phase constant. Thus we define the bandwidth of the amplifier as the $\Delta\omega$ which causes the deviation of $\Delta\beta$ to reduce the total gain of the amplifier by 3 db.

Assume the length of the traveling-wave structure is ℓ. Then the total gain for the most efficient conditions is

$$\alpha\ell = \frac{\omega_1^{\,2} C_1 L}{4\beta_1}\,\ell \quad \text{nepers} \tag{123}$$

The total gain of the amplifier with mismatch in phase constant $\Delta\beta$ is

$$\alpha'\ell = \left[\frac{\omega_1^{\,4} C_1^{\,2} L^2}{16\beta_1^{\,2}} - \left(\tfrac{1}{2}\Delta\beta\right)^2\right]^{\frac{1}{2}}\ell \qquad \text{nepers} \tag{124}$$

By the above bandwidth definition, we have

$$8.68 \times \left\{\frac{\omega_1^{\,2} C_1 L}{4\beta_1}\,\ell - \left[\frac{\omega_1^{\,4} C_1^{\,2} L^2}{16\beta_1^{\,2}} - \left(\tfrac{1}{2}\Delta\beta\right)^2\right]^{\frac{1}{2}}\ell\right\} = 3 \text{ d.b.} \tag{125}$$

assume

$$\Delta\beta^2 << \frac{\omega_1^{\,4} C_1^{\,2} L^2}{4\beta_1^{\,2}}$$

We then have from (125)

$$3 = 8.68\left\{\frac{\omega_1^{\,2} C_1 L}{4\beta_1}\,\ell - \frac{\omega_1^{\,2} C_1 L}{4\beta_1}\,\ell\left[1 - \frac{\Delta\beta^2}{\frac{\omega_1^{\,4} C_1^{\,2} L^2}{4\beta_1^{\,2}}}\right]^{\frac{1}{2}}\right\}$$

$$= 4.34\,\frac{\beta_1 \Delta\beta^2}{\omega_1^{\,2} C_1 L} \tag{126}$$

Hence

$$\Delta\beta = \sqrt{\frac{3}{4.34}} \sqrt{\frac{\omega_1{}^2 C_1 L}{\beta_1 \ell}} = 0.83 \sqrt{\frac{\beta_1 \eta}{\ell}} \tag{127}$$

The larger the $\Delta\beta$ the wider the bandwidth. Thus, for wide bandwidth operation, we would like to have η as large as possible. η is defined in Equation (43).

C. Noise Figure

Parametric circuits are expected to have excellent noise performance. The noise subject is a complex one and a complete and rigorous treatment of it is far beyond the intent of this section.

Let us investigate the noise figure of our particular circuit structure. For simplification, we consider the lossless case and assume $\Delta\beta=0$.

For the non-degenerate case, assume signal circuit (ω_1) is at temperature T_1 and idler circuit (ω_2) is at temperature T_2. Then the noise input of the signal and idler circuits are

$$N_{1i} = K\ T_1\ B \tag{128}$$

$$N_{2i} = K\ T_2\ B \tag{129}$$

where

N_{1i} and N_{2i} = noise inputs of signal and idler circuits

K = Boltzman constant

T_1 and T_2 = temperature in degree Kelvin of the signal and idler circuits

B = bandwidth in c.p.s.

For our traveling-wave parametric amplifier with length ℓ and gain constant α, the noise output N_{10} due to N_{1i} at ω_1 is

$$N_{10} = K\, T_1\, B\, e^{2\alpha \ell} \tag{130}$$

The above equation is obtained by neglecting the decaying wave in ω_1. Since noise energies observe the same Manley and Rowe relation, the noise output at ω_1 due to N_2 is

$$N_{20} = \frac{\omega_1}{\omega_2} K\, T_2\, B\, (e^{2\alpha \ell}-1) \tag{131}$$

Thus total noise output at ω_1 is

$$N_{TO} = K\, T_1\, B\, e^{2\alpha \ell} + \frac{\omega_1}{\omega_2} K\, T_2\, B\, (e^{2\alpha \ell}-1) \tag{132}$$

Noise figure is defined as the ratio of the noise to signal ratio at the output to that at the input. Assume

input signal energy = A

then,

output signal energy = A $e^{2\alpha \ell}$ (neglect the decaying wave)

Hence

$$\text{Noise figure } F = \frac{A/N_{1i}}{A\, e^{2\alpha \ell}/N_{TO}} \tag{133}$$

Equation (133) can be written as

$$F = 1 + \frac{\omega_1}{\omega_2}\frac{T_2}{T_1}\frac{(e^{2\alpha \ell}-1)}{e^{2\alpha \ell}} \tag{134}$$

From the above equation we know that in order to obtain a better noise figure the idler circuit temperature should be kept smaller than the signal circuit temperature. Nevertheless, for our degenerate circuitry (i.e. $\omega_1 \doteq \omega_2$, $T_1 \doteq T_2$), the noise figure is at most 3 db which is an excellent noise figure.

D. The Effect Due to Resistive Loss of the Signal Line

The signal coils for this type of amplifier are wound with very fine wire which has considerable resistance, thus it is appropriate to investigate the effect of this resistive loss on the amplifier performance.

Fig. 3 can be redrawn as shown in Fig. 4 with R_1 representing the series resistance of the signal coils and R_2 representing the resistance in the idler circuits.

Let $V_1(z,t)$, $V_2(z,t)$, $I_1(z,t)$ and $I_2(z,t)$ be the voltages and currents of signal and idler circuits. We then can write

$$\frac{\partial V_1(z,t)}{\partial z} + L_1 \frac{\partial I_1(z,t)}{\partial t} + R_1 I_1(z,t) + \frac{\partial}{\partial t}[L(z,t) I_2(z,t)] = 0 \qquad (135)$$

$$\frac{\partial I_1(z,t)}{\partial z} + C_1 \frac{\partial V_1(z,t)}{\partial t} = 0 \qquad (136)$$

$$\frac{\partial V_2(z,t)}{\partial z} + L_2 \frac{\partial I_2(z,t)}{\partial t} + R_2 I_2(z,t) + \frac{\partial}{\partial t}[L(z,t) I_1(z,t)] = 0 \qquad (137)$$

$$\frac{\partial I_2(z,t)}{\partial z} + C_2 \frac{\partial V_2(z,t)}{\partial t} = 0 \qquad (138)$$

After combining Equation (135) with Equation (136) and Equation (137) with Equation (138), we obtain

$$\frac{\partial^2 I_1(z,t)}{\partial z^2} = C_1 L_1 \frac{\partial^2 I_1(z,t)}{\partial t^2} + C_1 R_1 \frac{\partial I_1(z,t)}{\partial t} + C_1 \frac{\partial^2}{\partial t^2}[L(z,t) I_2(z,t)] \qquad (139)$$

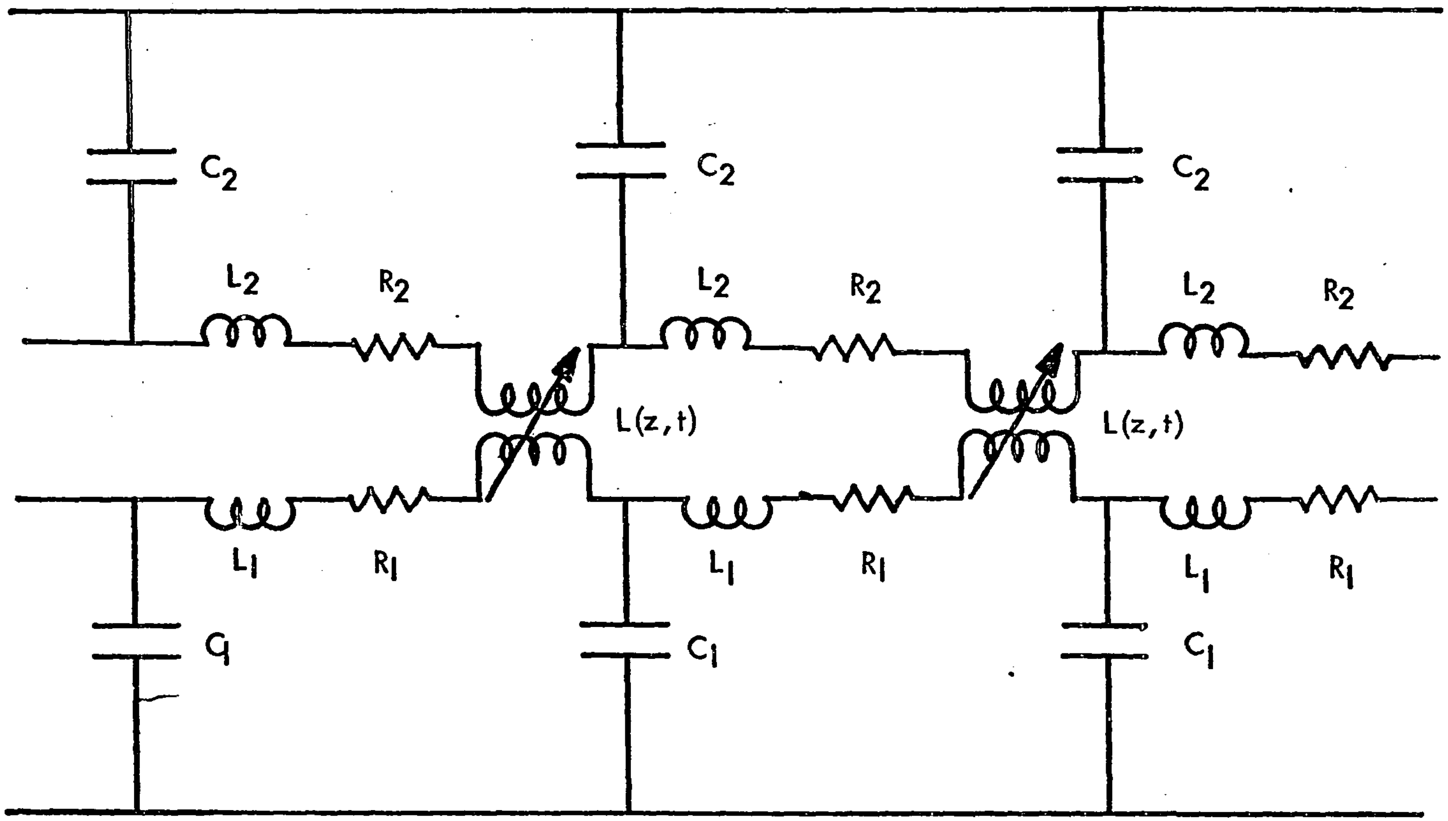

Fig. 4. Propagating structure with resistive loss

$$\frac{\partial^2 I_2(z,t)}{\partial z^2} = C_2L_2 \frac{\partial^2 I_2(z,t)}{\partial t^2} + C_2R_2 \frac{\partial I_2(z,t)}{\partial t} + C_2 \frac{\partial^2}{\partial t^2}[L(z,t)I_1(z,t)] \tag{140}$$

As before, put

$$I_1(z,t) = I_1(z)e^{j\omega_1 t} + I_1^*(z)e^{-j\omega_1 t}$$

$$I_2(z,t) = I_2(z)e^{j\omega_2 t} + I_2^*(z)e^{-j\omega_2 t}$$

$$L(z,t) = \frac{1}{2}[L(z)e^{j\omega t} + L^*(z)e^{-j\omega t}] \tag{141}$$

Substitute relations (141) into Equations (139) and (140). After some manipulation we can get two pairs of simultaneous differential equations. The pair of equations concerning $I_1(z)$ and $I_2^*(z)$ are

$$\frac{\partial^2 I_1(z)}{\partial z^2} = -\omega_1^2C_1L_1I_1(z) + j\omega_1C_1R_1I_1(z) - \frac{1}{2}C_1\omega_1^2L(z)I_2^*(z) \tag{142}$$

$$\frac{\partial^2 I_2^*(z)}{\partial z^2} = -\omega_2^2C_2L_2I_2^*(z) - j\omega_2C_2R_2I_2^*(z) - \frac{1}{2}C_2\omega_2^2L^*(z)I_1(z) \tag{143}$$

now put

$$I_1(z) = A_1(z)e^{-j\beta_1 z} \quad , \quad I_2(z) = A_2(z)e^{j\beta_2 z}$$

$$L(z) = Le^{-j\beta z} \quad , \quad L^*(z) = Le^{j\beta z} \tag{144}$$

Substituting relations (144) into Equations (142) and (143), we get

$$\frac{\partial^2 A_1(z)}{\partial z^2} - 2j\beta_1 \frac{\partial A_1(z)}{\partial z} - \beta_1^2A_1(z) = -\omega_1^2C_1L_1A_1(z) + j\omega_1C_1R_1A_1(z) - \frac{1}{2}C_1\omega_1^2LA_2^*(z) \tag{145}$$

$$\frac{\partial^2 A_2^*(z)}{\partial z^2} + 2j\beta_2 \frac{\partial A_2^*(z)}{\partial z} - \beta_2^2 A_2^*(z) = -\omega_2^2 C_2 L_2 A_2^*(z) - j\omega_2 C_2 R_2 A_2^*(z)$$

$$- \frac{1}{2} C_2 \omega_2^2 L A_1(z) \qquad (146)$$

After some cancellation and simplification, Equations (145) and (146) can be reduced to

$$\begin{cases} (-2j\beta_1 \frac{\partial}{\partial z} - j\omega_1 C_1 R_1) A_1(z) + \frac{1}{2} C_1 \omega_1^2 L A_2^*(z) = 0 & (147) \\ (2j\beta_2 \frac{\partial}{\partial z} + j\omega_2 C_2 R_2) A_2^*(z) + \frac{1}{2} C_2 \omega_2^2 L A_1(z) = 0 & (148) \end{cases}$$

Combining the above two simultaneous equations into a single differential equation with dependent variable $A_1(z)$, we have

$$4\beta_1\beta_2 \frac{\partial^2 A_1(z)}{\partial z^2} + 2j(\omega_1 C_1 R_1 \beta_2 + \omega_2 C_2 R_2 \beta_1) \frac{\partial A_1(z)}{\partial z}$$

$$+ (\omega_1 \omega_2 C_1 C_2 R_1 R_2 - \frac{1}{4} C_1 C_2 \omega_1^2 \omega_2^2 L^2) A_1(z) = 0 \qquad (149)$$

The solution of Equation (149) is of the form

$$A_1(z) = a_1 e^{r_1 z} + a_2 e^{r_2 z} \qquad (150)$$

where

$$r_{1,2} = -\frac{j}{4\beta_1\beta_2}(\omega_1 C_1 R_1 \beta_1 + \omega_2 C_2 R_2 \beta_2)$$

$$\pm \frac{1}{2}\left[-\left(\frac{\omega_1 C_1 R_1 \beta_1 + \omega_2 C_2 R_2 \beta_2}{2\beta_1\beta_2}\right)^2 - \frac{1}{\beta_1\beta_2}(\omega_1\omega_2 C_1 C_2 R_1 R_2 - \frac{1}{4} C_1 C_2 \omega_1^2 \omega_2^2 L^2)\right]^{\frac{1}{2}}$$

$$(151)$$

For our degenerate case, we have

$$\omega_1 = \omega_2 \quad , \quad C_1 = C_2 \quad , \quad \beta_1 = \beta_2 \quad , \quad R_1 = R_2$$

then Equation (151) becomes

$$r_{1,2} = -\frac{j}{2\beta_1}(\omega_1 C_1 R_1) \pm \frac{1}{2}[-(\frac{\omega_1 C_1 R_1}{2\beta})^2 - \frac{\omega_1{}^2 C_1{}^2 R_1{}^2}{\beta_1{}^2} + 4\zeta^2]^{\frac{1}{2}}$$

$$= + j\beta_r \pm \alpha_r \tag{152}$$

The amplitude of the signal current can then be written as

$$A_1(z) = e^{j\beta rz}[a_1 e^{\alpha rz} + a_2 e^{-\alpha rz}] \tag{153}$$

where

$$\beta_r = -\frac{\omega_1 C_1 R_1}{2\beta_1} \quad , \quad \alpha_r = [\zeta^2 - \frac{5}{16}(\frac{\omega_1 C_1 R_1}{\beta_1})^2] \tag{154}$$

From Equation (153) we see that an additional phase constant β_r is introduced by the lossy coils. In addition, the gain constant α_r is reduced due to the resistance R_1. Thus the resistive coils complicate an optimum design for the amplifier.

E. Initial Phase Angles Consideration

In the following analysis we shall determine the relationship of the initial phase angles of the signal, idler and pump waves to the gain constant α. Finally we shall briefly consider the effect of pump line losses to the signal gain.

The degenerate signal wave propagating structure is shown in Fig. 5. The voltage and current relations for this structure are

$$-\frac{\partial i}{\partial z} = C\frac{\partial v}{\partial t} \tag{155}$$

$$-\frac{\partial v}{\partial z} = \frac{d(Li)}{dt} \tag{156}$$

where i and v are both functions of time t and distance z.

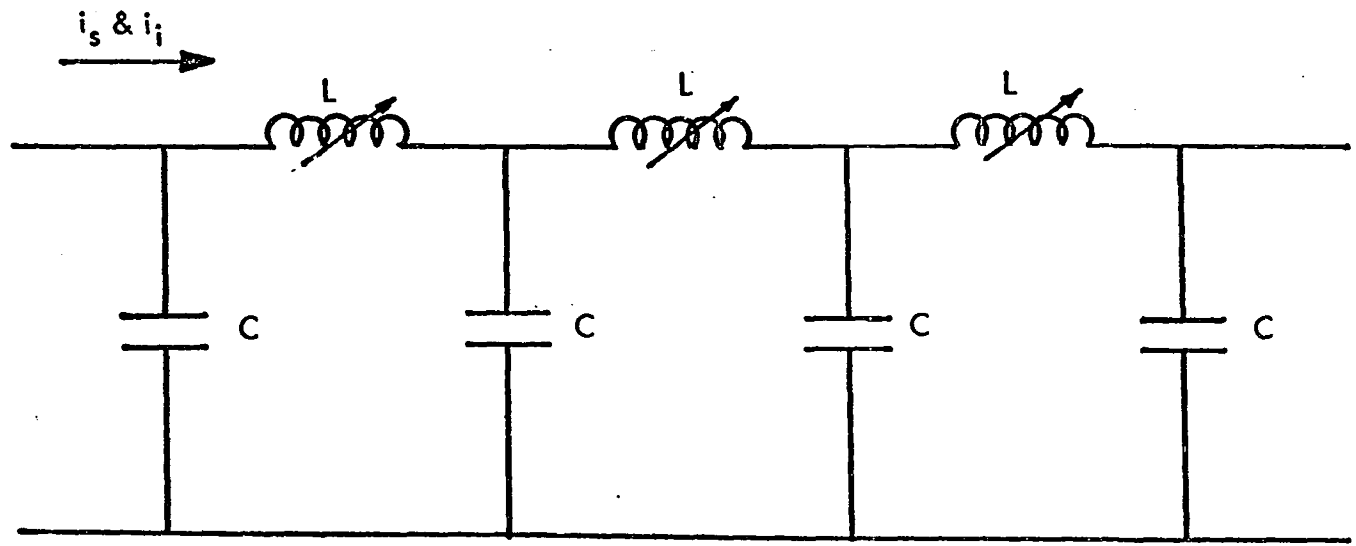

Fig. 5. Degenerate wave propagating structure

From Equations (155) and (156) we can get

$$\frac{\partial^2 i}{\partial z^2} = C\,\frac{\partial^2 (iL)}{\partial t^2} \tag{157}$$

now put

$$L = L_o(1+\delta i_p) \tag{158}$$

$$= L_o(1+\delta I_p \cos\omega t)$$

Compared to Equation (42), we know that

$$\delta = \frac{n}{I_p} \tag{159}$$

then Equation (157) becomes

$$\frac{\partial^2 i}{\partial z^2} = CL_o\,\frac{d^2 i}{dt^2} + \delta CL_o\,\frac{d^2(i_p i)}{dt^2} \tag{160}$$

As before, assume the propagating structure can only pass signal and idler waves. Thus all the higher frequency waves are attenuated, so we assume

$$i = I_1\cos(\omega_1 t-\beta_1 z+\sigma_1) + I_2\cos(\omega_2 t-\beta_2 z+\sigma_2) \tag{161}$$

$$i_p = I_p\cos(\omega_p t-\beta_p z+\sigma_p) \tag{162}$$

where

i_1 = signal current

i_2 = idler current

i_p = pump current

σ's are initial phase angles .

Thus we have

$$i_p\ i = I_1 I_p \cos(\omega_1 t-\beta_1 z+\sigma_1)\cos(\omega_p t-\beta_p z+\sigma_p)$$

$$+ I_2 I_p \cos(\omega_2 t-\beta_2 z+\sigma_2)\cos(\omega_p t-\beta_p z+\sigma_p)$$

$$= \frac{I_1 I_p}{2}[\cos(\omega_1 t+\omega_p t-\beta_1 z-\beta_p z+\sigma_1+\sigma_p)$$

$$+ \cos(\omega_p t-\omega_1 t-\beta_p z-\beta_1 z-\sigma_p-\sigma_1)]$$

$$+ \frac{I_2 I_p}{2}[\cos(\omega_2 t+\omega_p t-\beta_p z-\beta_2 z+\sigma_2+\sigma_p)$$

$$+ \cos(\omega_p t-\omega_2 t-\beta_p z+\beta_2 z+\sigma_p-\sigma_2)] \qquad (163)$$

By discarding the higher frequency components, Equation (163) becomes

$$i_p\ i = \frac{I_1 I_p}{2}\cos(\omega_2 t-\beta_2 z+\sigma_p-\sigma_1)$$

$$+ \frac{I_2 I_p}{2}\cos(\omega_1 t-\beta_1 z+\sigma_p-\sigma_2) \qquad (164)$$

Substituting Equations (161) and (164) into Equation (160), we get

$$\frac{\partial^2}{\partial z^2}[I_1\cos(\omega_1 t-\beta_1 z+\sigma_1) + I_2\cos(\omega_2 t-\beta_2 z+\sigma_2)]$$

$$= CL_o\,\frac{\partial^2}{\partial t^2}[I_1\cos(\omega_1 t-\beta_1 z+\sigma_1) + I_2\cos(\omega_2 t-\beta_2 z+\sigma_2)]$$

$$+ CL_o\,\frac{\partial^2}{\partial t^2}[\frac{I_1 I_p}{2}\cos(\omega_2 t-\beta_2 z+\sigma_p-\sigma_1) + \frac{I_2 I_p}{2}\cos(\omega_1 t-\beta_1 z+\sigma_p-\sigma_2)]$$

$$(165)$$

Since I_1 and I_2 are functions of z only, the left side of Equation (165) can be expanded as

$$\frac{\partial^2}{\partial z^2}[I_1\cos(\omega_1 t-\beta_1 z+\sigma_1) + I_2\cos(\omega_2 t-\beta_2 z+\sigma_2)]$$

$$= \frac{d^2 I_1}{dz^2}\cos\theta_1 - I_1\beta_1{}^2\cos\theta_1 + 2\beta_1\sin\theta_1\frac{dI_1}{dz}$$

$$+ 2\beta_2\sin\theta_2\frac{dI_2}{dz} - I_2\beta_2{}^2\cos\theta_2 + \frac{d^2 I_2}{dz^2}\cos\theta_2 \qquad (166)$$

where

$$\theta_n = \omega_n t - \beta_n z + \sigma_n \qquad n=1,2,3\ . \qquad (167)$$

Right hand side of Equation (165) becomes

$$CL_o\frac{\partial^2}{\partial t^2}[I_1\cos\theta_1 + I_2\cos\theta_2)] + \delta CL_o\frac{\partial^2}{\partial t^2}[\frac{I_1 I_p}{2}\cos(\theta_2-\psi)$$

$$+ \frac{I_2 I_p}{2}\cos(\theta_1-\psi)]$$

$$= -CL_o\omega_1{}^2 I_1\cos\theta_1 - CL_o\omega_2{}^2 I_2\cos\theta_2 - \delta CL_o\frac{I_1 I_p}{2}\omega_2{}^2\cos(\theta_2-\psi)$$

$$- \delta CL_o\frac{I_2 I_p}{2}\omega_1{}^2\cos(\theta_1-\psi) \qquad (168)$$

where

$$\psi = \sigma_1 + \sigma_2 + \sigma_3 \qquad (169)$$

From Equations (165), (166) and (168), we have

$$\frac{d^2I_1}{dz^2}\cos\theta_1 - I_1\beta_1{}^2\cos\theta_1 + 2\beta_1\sin\theta_1\frac{dI_1}{dz} + 2\beta_2\sin\theta_2\frac{dI_2}{dz}$$

$$- I_2\beta_2{}^2\cos\theta_2 + \frac{d^2I_2}{dz^2}\cos\theta_2 = - CL_o\omega_1{}^2I_1\cos\theta_1$$

$$- CL_o\omega_2{}^2I_2\cos\theta_2 - \delta CL_o\frac{I_1I_p}{2}\omega_2{}^2\cos(\theta_2-\psi) - \delta CL_o\frac{I_2I_p}{2}\omega_1{}^2\cos(\theta_1-\psi) \tag{170}$$

We can get two independent equations from Equation (170) for ω_1 and ω_2 respectively. For ω_1, we have

$$2\beta_1\sin\theta_1\frac{dI_1}{dz} + \cos\theta_1\frac{d^2I_1}{dz^2} - I_1\beta_1{}^2\cos\theta_1$$

$$= - CL_o\omega_1{}^2I_1\cos\theta_1 - \delta CL_o\frac{I_2I_p}{2}\omega_1{}^2\cos(\theta_1-\psi) \tag{171}$$

For ω_2, we have

$$2\beta_2\sin\theta_2\frac{dI_2}{dz} - I_2\beta_2{}^2\cos\theta_2 + \cos\theta_2\frac{d^2I_2}{dz^2}$$

$$= -CL_o\omega_2{}^2I_2\cos\theta_2 - \delta CL_o\frac{I_1I_p}{2}\omega_2{}^2\cos(\theta_2-\psi) \tag{172}$$

By separating the phase quadrature components, we get from Equation (171)

$$4\beta_1\frac{dI_1}{dz} = - \delta CL_oI_2I_p\omega_1{}^2\sin\psi \tag{173}$$

$$\frac{d^2I_1}{dz^2} = I_1\beta_1{}^2 - CL_o\omega_1{}^2I_1 - \delta CL_o\frac{I_2I_p}{2}\omega_1{}^2\cos\psi \tag{174}$$

Similarly, from Equation (172), we get

$$4\beta_2\frac{dI_2}{dz} = -\delta CL_oI_1I_p\omega_2{}^2\sin\psi \tag{175}$$

$$\frac{d^2I_2}{dz^2} = I_1\beta_1{}^2 - CL_o\omega_1{}^2I_1 - \delta CL_o \frac{I_2I_p}{2} \omega_1{}^2\cos\psi \tag{176}$$

Since

$$\beta_1 \frac{dI_1}{dz} >> \frac{d^2I_1}{dz^2} \quad \text{and} \quad \beta_2 \frac{dI_2}{dz} >> \frac{d^2I_2}{dz^2}$$

we need only to consider Equations (173) and (175).

Differentiating Equation (173) with respect to z, we get

$$\frac{d^2I_1}{dz^2} = -\frac{\delta CL_o}{4\beta_1} I_p\omega_1{}^2\sin\psi \frac{dI_2}{dz} \tag{177}$$

Substituting Equation (175) into Equation (177), we have

$$\frac{d^2I_1}{dz^2} = \frac{1}{16\beta_1\beta_2} (\delta CL_oI_p\sin\psi\omega_1\omega_2)^2 I_1 \tag{178}$$

Thus, the gain constant α is

$$\alpha = \frac{\delta CL_oI_p\sin\psi\omega_1\omega_2}{4\sqrt{\beta_1\beta_2}} \tag{179}$$

Equation (179) shows that the gain constant changes periodically as function of ψ. At the optimum condition (i.e., $\sin\psi=1$), the expression for gain constant is the same as in previous analysis. Thus the solution of Equation (179) can be expressed as

$$I_1 = a_1 \cosh \alpha z + a_2 \sinh \alpha z \tag{180}$$

and the initial condition is

$$I_1(z) = I_{10} \quad \text{at} \quad z=0 .$$

Therefore the signal current may be written as

$$I_1 = I_{10} \cosh \alpha z \quad . \tag{181}$$

From the Manley and Rowe relation, we have

$$\frac{P_1}{\omega_1} = \frac{P_2}{\omega_2} \tag{182}$$

thus the idler current becomes

$$|I_2| = \sqrt{\frac{\omega_2}{\omega_1}}\ |I_1| \quad .$$

The initial condition for I_2 is

$$I_2(o) = 0 \quad .$$

Thus the solution for idler current is

$$I_2(z) = \sqrt{\frac{\omega_2}{\omega_1}}\, I_{10} \sinh \alpha z \quad . \tag{183}$$

Equations (181) and (183) are in agreement with Equations (80) and (81).

F. The Effect of Pump Line Loss

Now we shall consider the effect of the pump line loss on the signal current. Assume the attenuation of the pump line current is α_p which is mainly due to the series resistance of the pump line. Thus, we have

$$I_p = I_{po}\, e^{-\alpha_p z} \tag{184}$$

Then the gain constant of Equation (179) becomes

$$\alpha = \frac{\delta C L_o \sin\psi \omega_1 \omega_2}{4\sqrt{\beta_1 \beta_2}}\ I_{po}\, e^{-\alpha_p z} = \alpha_o\, e^{-\alpha_p z} \tag{185}$$

where

$$\alpha_o = \frac{\delta C L_o \omega_1 \omega_2 \, I_{po} \, \sin\psi}{4 \sqrt{\beta_1 \beta_2}}$$

From Equation (181), we get

$$I_1 = I_{10} \cosh \alpha z = I_{10} \cosh \int_o^z \alpha dz$$

$$= I_{10} \cosh \int_o^z \alpha_o \, e^{-\alpha_p z} \, dz = I_{10} \cosh \frac{\alpha_o}{\alpha_p} (1-e^{-\alpha_p z}) \qquad (186)$$

Equation (186) shows the extent to which the pump line attenuation limits the signal current gain. This interesting effect is shown in Fig. 6. At $n = \frac{\alpha_p}{\alpha} = 0.5$ the gain is limited to around 10 db instead of increasing exponentially. Thus in order to obtain higher gain, it is important to keep the pump line loss as small as possible.

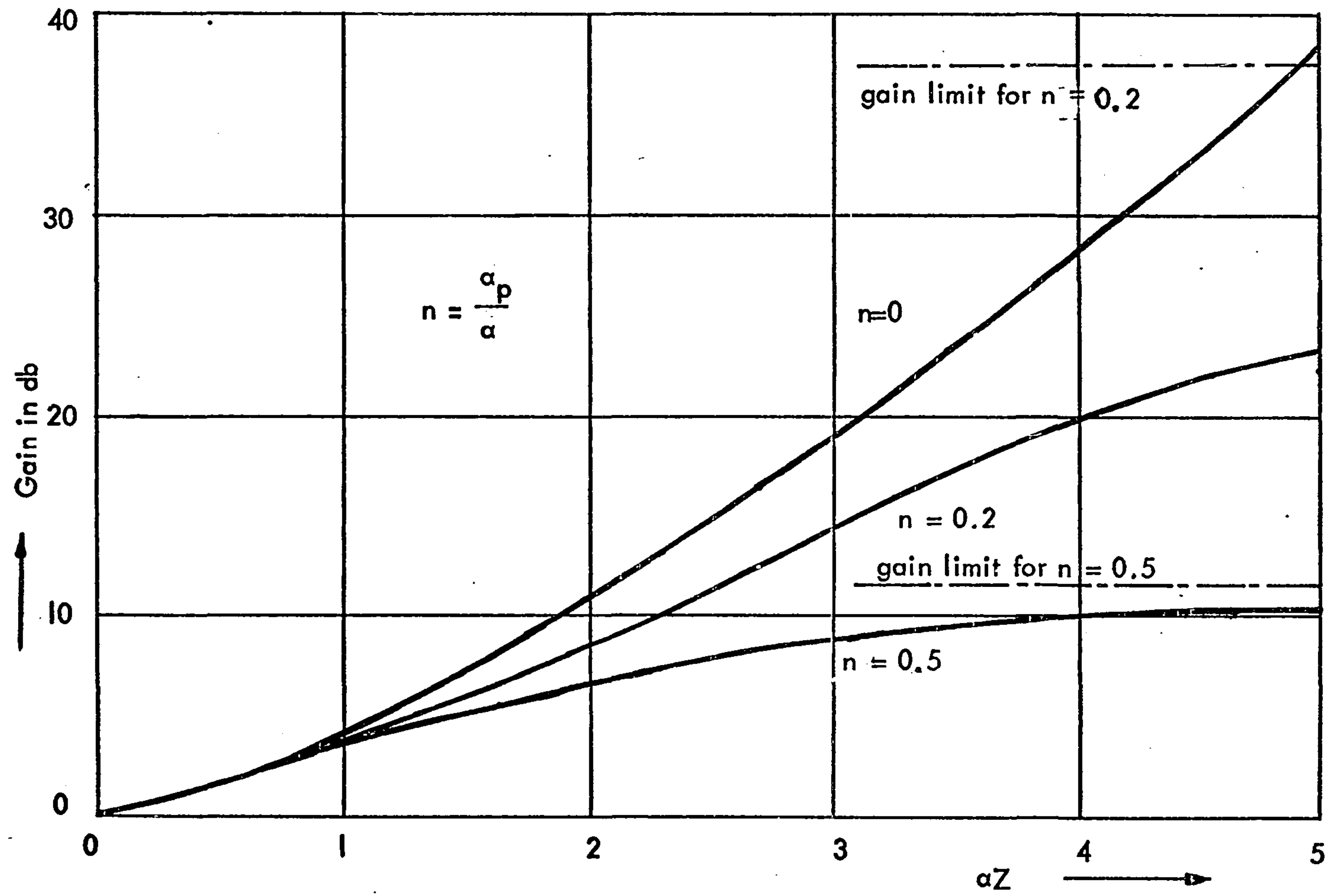

Fig. 6. The effect of pump line loss

IV. EXPERIMENTAL WORK

Before entering into the design detail of the traveling-wave parametric amplifier, a formula for calculating the pump line inductance must be derived.

The flux density of the thin film in easy direction due to the magnetization vector $\vec{M}$ is (see Fig. 2)

$$B_y = M \cos\theta \tag{187}$$

Since θ is small, $\cos\theta$ is equal to unity. Thus the flux linkage per unit length of the pump line is

$$\lambda = Mt \tag{188}$$

where

t = the thickness of the thin film.

Thus the inductance per unit length due to magnetic thin film is

$$L_{pf} = \frac{\lambda}{I_p + I_b} = \frac{Mt}{I_p + I_b} \tag{189}$$

The inductance per unit length of a conductor with radius r_o is

$$L_{pa} = \frac{\mu_o}{2\pi} \ln \frac{2h}{r_o} \tag{190}$$

where

h = distance between the conductor and the ground.

Therefore the total pump line inductance per unit length may be expressed as

$$L_p = L_{pf} + L_{pa}$$

$$= \frac{Mt}{I_p + I_b} + \frac{\mu_o}{2\pi} \ln \frac{2h}{r_o} \tag{191}$$

From Equations (43), (38) and (39), the inductance of the signal line may be expressed as

$$L_1 = N^2\mu_o A + \frac{N^2MA_f}{H_K+H_b} \tag{192}$$

The following parameters are used for calculating the inductances:

M	10000 gauss	1 weber/m^2
H_K	2.5 oe	200 a-t/m
t	15000 Å	1.5 μ
H_b	2.5 oe	200 a-t/m
r_o	2.5 mils	63.4 μ
h	2.5 cm	0.025 m

The calculated pump line inductance is 0.129 μh. The pump line generator output impedance is 50 ohms. Therefore the pump line is designed with a 50 ohm characteristic impedance. That is

$$C_p = \frac{L_p}{z_{op}^2} = 51.5 \text{ μμf}$$

Therefore the pump line capacitance is chosen to be 50 μμf.

The calculated signal line inductance with number of turns N=270 is 5.73 μh per section. The phase constants of pump and signal lines per section of signal line are

$$\beta_p = 2\omega_p \sqrt{L_pC_p}$$
$$\beta_1 = \omega_1 \sqrt{L_1C_1} \tag{193}$$

The very important condition that must be satisfied by the phase constants is

$$\beta_p = \beta_1 + \beta_2 \tag{194}$$

For the degenerate traveling-wave parametric amplifier, we take

$$\beta_1 = \beta_2$$

$$\omega_1 = \omega_2$$

$$\omega_p = 2\omega_1 \tag{195}$$

Substituting Equations (193) and (195) into relation (194), we have

$$2\sqrt{L_p C_p} = \sqrt{L_1 C_1} \tag{196}$$

where

$$L_p = 0.129 \ \mu h$$

$$C_p = 50 \ \mu\mu f$$

$$L_1 = 5.73 \ \mu h$$

thus

$$C_1 = \frac{4L_p C_p}{L_1} = 4.5 \ \mu\mu f$$

The circuit of Fig. 7 was constructed using the above values for $C_1 = C_s$, C_p, $L_1 = L_s$, and L_p. R_{Lp} and R_{Ls} are load resistances for the pump line and signal line respectively. E is the variable d.c. source for supplying the bias current of the pump line. L_{c1} and L_{c2} are chokes to prevent pump current entering into d.c. bias circuit. V_p and V_s are pump source and signal generator respectively. The transformer T_s is used to match the low signal generator output impedance and the high characteristic impedance of the signal line.

A photograph of the test model is shown in Fig. 8. A Collins type 32S-1 transmitter operating at about 14 mc/s was used as the pump source. A

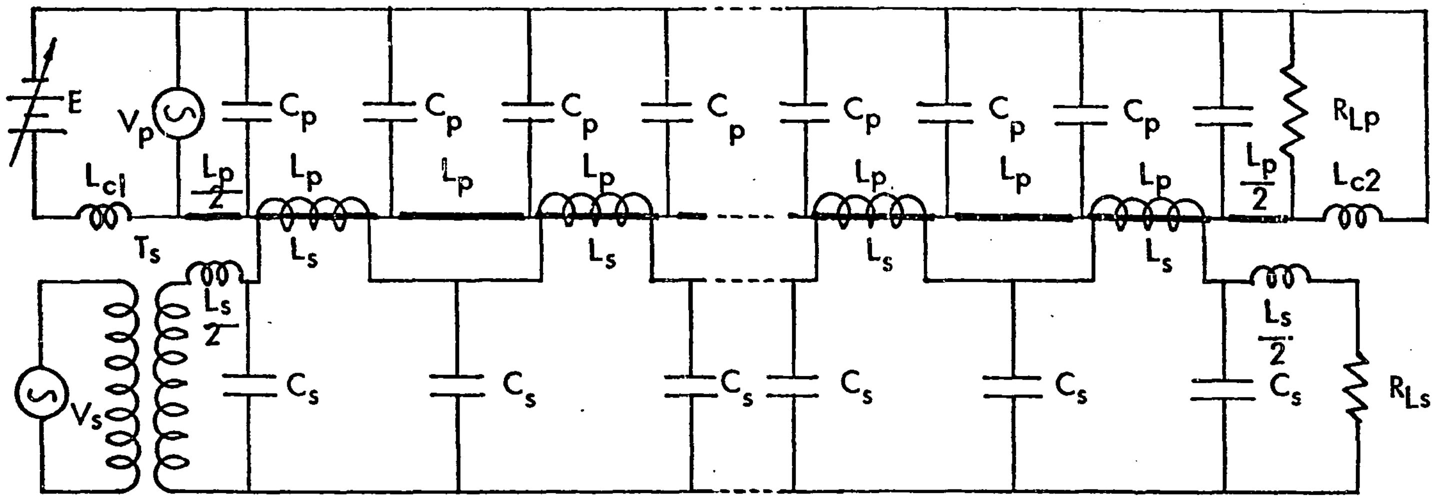

Fig. 7. Experimental Arrangement

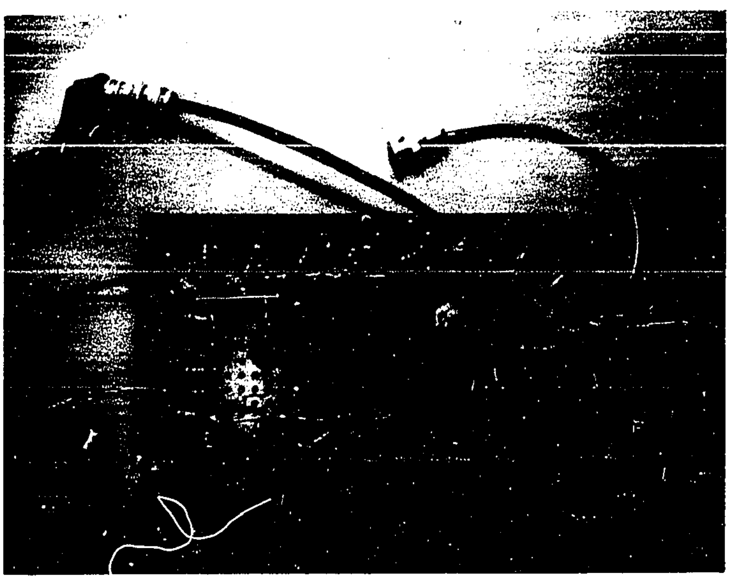

Fig. 8. The Test Model

signal of about 7 mc/s was connected to the signal line. The voltage distribution along the structure was measured by means of a Hewlett-Packard Model 150A oscilloscope. It was apparent that the signal did not grow exponentially along the structure; rather, the signal actually appeared to vary periodically along the line. This periodic variation is maily due to the mismatch of the phase constants of pump line and signal line. It could be caused by the following factors:

(1) The capacitors used for constructing the propagating structures are of 20% accuracy.

(2) The calculated inductances for the pump and signal lines may be different from the true values due to the deviations of the thin film thickness and other parameters from the specified values.

(3) As shown in Chapter III, the resistive signal line loss changes the phase constant and the gain constant also.

Thus the value of $\frac{\Delta\beta}{2}$ may easily exceed that of the gain factor ζ. In such a case, instead of the exponentially growing signal current of Equation (80), Equation (117) will be the appropriate expression for the signal current. Gain of the signal will be produced when $\frac{\Delta\beta}{2}$ and ζ are almost equal. If $[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}}$ is very small in comparison with $\frac{\Delta\beta}{2}$, Equation (117) can be written as

$$I_1(z,t) = \frac{\zeta^2 I_{10}}{(\frac{\Delta\beta}{2})^2} \cos\,[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}} z \cos\,(\omega_1 t - \beta_1' z + \sigma)$$

$$- \frac{\zeta^2 I_{10}}{\frac{\Delta\beta}{2}[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}}} \sin\,[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}} z \sin\,(\omega_1 t - \beta_1' z + \sigma) \tag{197}$$

The normalized expression for signal gain is

$$\left|\frac{I_1(z)}{I_{10}}\right| = \cos\,[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}} z + \frac{\zeta}{[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}}} \sin\,[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}} z$$

Fig. 9 is a comparison of calculated gain and experimental gain versus $[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}} z$ in electrical angle. The value of $[(\frac{\Delta\beta}{2})^2 - \zeta^2]^{\frac{1}{2}}$ is found to be 0.0945 rad/cm. The arrows on the figure represent the extent of the scattered data which were measured along the signal line. The solid line represents best fit. The dashed line is the calculated curve.

As shown in Equation (179) of Chapter III the gain constant is a sine function of the initial phase angles of pump, signal and idler waves. The initial phase angle relationship of the present pump source and signal generator changes continuously. Thus, at each fixed point of the signal line structure, the gain of the signal is observed to have a sinusoidal variation. Therefore, with constant input signal frequency, it is very important to maintain a constant pump frequency.

The gain of the amplifier is a function of ζ which is in turn

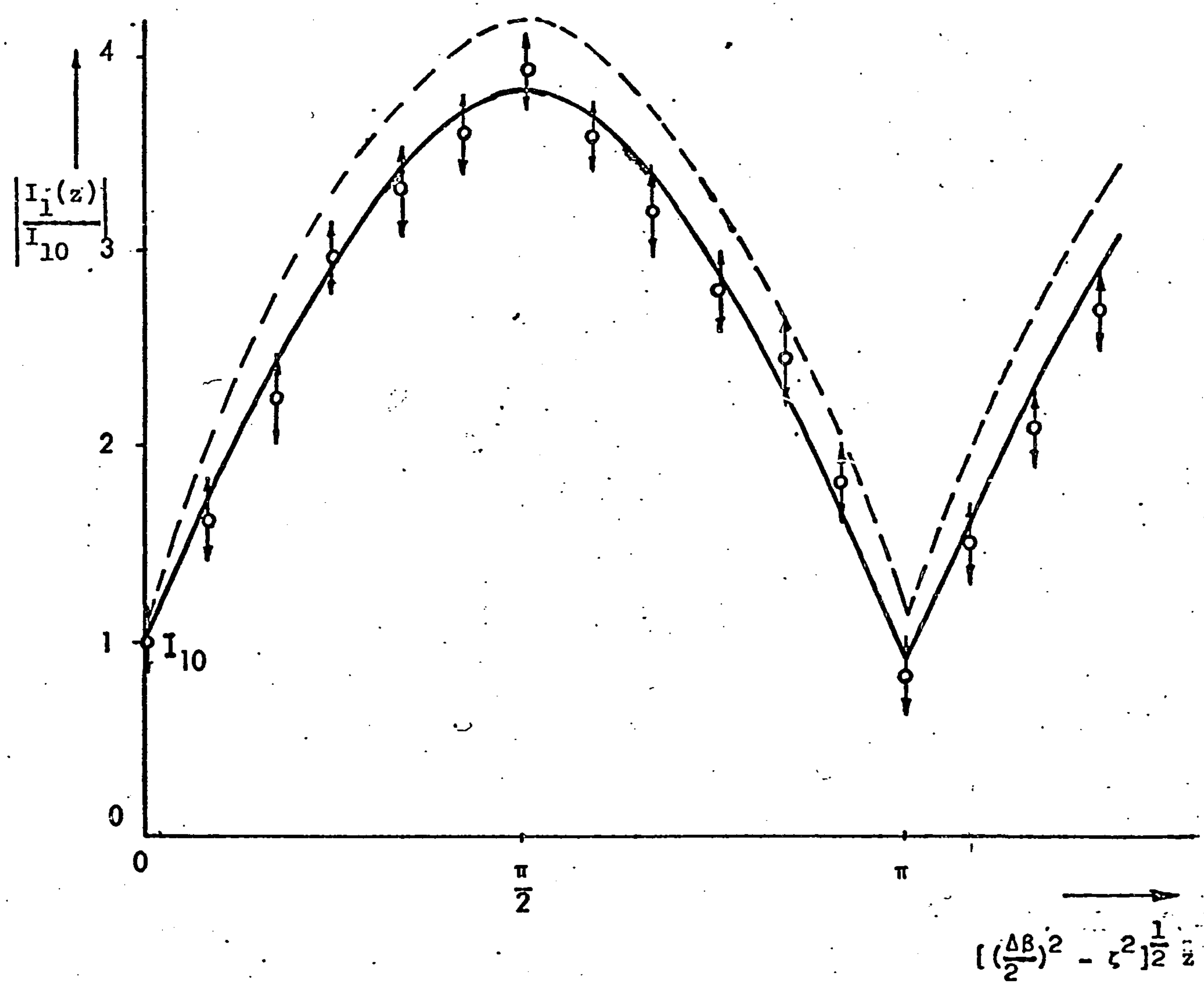

Fig. 9. Periodic variation of the gain of signal current

proportional to the variable inductance L. From Equations (43) and (40), we know that L is proportional to a, which in turn is proportional to the pump field H_p. Thus we observe a substantial increase of the gain of the amplifier by increasing the pump power. A smaller amount of decrease in gain by increasing the bias field is also observed. This is because H_b is in the denominator of Equation (40) and its value is comparable to that of H_K. The bandwidth of the experimental amplifier was measured to be around 15%.

V. SUMMARY

The purpose of this investigation was to study analytically and experimentally the wave propagation in the periodic traveling-wave structure using a thin magnetic film as a time-varying inductive parameter. The ferromagnetic thin film was electro-deposited on 5 mil diameter beryllium-copper wire with the easy direction circumferential to the axis of the wire and the hard direction parallel to the axis of the wire. This thin magnetic film can be used as time-varying inductors which couples the pump power into the signal line, thus producing parametric signal gain.

The analysis has shown that an exponentially growing signal wave is possible under certain favorable conditions. The mismatch of the phase constants of pump line and signal line reduces the gain constant of the signal. For larger mismatch of the phase constants, the signal gain varies periodically instead of exponentially. Bandwidth and noise figure of the thin film traveling-wave parametric amplifier are briefly discussed. It was further shown that the resistive loss of the signal line reduces the gain constant and also changes the phase constant of the signal line, thus complicating optimum amplifier design. It has also been shown that the gain constant changes periodically as function of initial phase angles of pump, signal and idler waves. The limitation of the gain of the amplifier due to the pump line loss was also considered.

An experimental thin film traveling-wave parametric amplifier was constructed to investigate its performance. The periodically varying signal gain was observed along the signal line which is mainly due to the mismatch of the phase constants of pump and signal lines. The continuous changing of the initial phase angle relationship between signal and pump caused the sinusoidal variation of the gain constant of the signal. As predicted in the theory, the gain of the amplifier increases with increasing pump power. The effect of bias field of the thin film was also observed as predicted. The bandwidth of the experimental amplifier was measured to be around 15%.

VI. BIBLIOGRAPHY

1. Hartley, R. V. L. Oscillations in systems with nonlinear reactance. Bell System Technical Journal. 15: 424-440. 1936.

2. van der Ziel, A. On the mixing properties of nonlinear capacitances. J. Appl. Phys. 19: 999-1006. 1948.

3. Manley, J. M. and Rowe, H. E. Some general properties of non-linear elements. I. General energy relations. Proc. Inst. Radio Engrs. 44: 904-913. 1956.

4. Suhl, H. Proposal for a ferromagnetic amplifier in the microwave range. Phys. Rev. 106: 384-385. 1957.

5. Weiss, M. T. A solid-state microwave amplifier and oscillator using ferrites. Phys. Rev. 107: 317. 1957.

6. Heffner, H. and Kotzebue, K. Experimental characteristics of a microwave parametric amplifier using semiconductor diodes. Proc. Inst. Radio Engrs. 46: 1301. 1958.

7. Chang, K. K. N. and Bloom, S. Parametric amplifier using lower-frequency pumping. Proc. Inst. Radio Engrs. 46: 1383-1386. 1958.

8. Read, A. A. and Pohm, A. V. Magnetic film parametric amplifiers. Proc. of the National Electronics Conference. (1959) 15: 65-78. 1960.

9. Read, A. A. and Samuels, R. L. Thin film balanced modulators. Electronics. 33, No. 9: 78-80. Feb. 1960.

10. Pohm, A. V., Read, A. A., Stewart, R. M., and Schauer, R. F. Operation of magnetic thin film parametrons in the 100-500 Mc regions. J. Appl. Phys. 31, No. 5: 119S-120S. 1960.

11. Little, A. G. A wide-band single-diode parametric amplifier using filter technique. Proc. Inst. Radio Engrs. 49: 821-822. 1961.

12. Cullen, A. L. A traveling-wave parametric amplifier. Nature. 181: 332. 1958.

13. Bell, C. V. and Wade, G. Circuit consideration in traveling-wave parametric amplifiers. Inst. of Radio Engrs. WESCON Convention Record 1959, Part 2: 75-83. 1959.

14. Engelbrecht, R. S. A low-noise nonlinear reactance traveling-wave amplifier. Proc. of Inst. of Radio Engrs. 46: 1655. 1958.

15. Lombardo, P. P. and Saard, E. W. Low-frequency prototype traveling-wave reactance amplifier. Proc. of Inst. Radio Engrs. 47: 995-996. 1959.

16. Tien, P. K. and Suhl, H. A traveling-wave ferromagnetic amplifier. Proc. of Inst. Radio Engrs. 46: 700-706. 1958.

17. Kurokawa, K. and Hamasaki, J. Mode theory of lossless periodically distributed parametric amplifiers. Inst. of Radio Engrs. Trans. on Microwave Theory and Techniques. 7: 360-365. 1959.

18. Mavaddat, R. and Hyde, F. J. Investigation of an experimental traveling-wave parametric amplifier. Proc. of Instn. Elect. Engrs. 109B: 405-413. 1962.

19. Louisell, W. H. Coupled Mode and Parametric Electronics. New York, N.Y., John Wiley and Sons, Inc. 1960.

VII. ACKNOWLEDGEMENTS

The author wishes to express his appreciation to his Major Professor Dr. R. M. Stewart for his guidance and helpful suggestions during the preparation of the manuscript. The author also wishes to thank Dr. R. E. Post and Dr. A. V. Pohm for their many suggestions during the project.

05

465